MATH 430 Problem Set 1

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Problem 1. Show that 16 divides $n^4 + 4n^2 + 11$ for every odd integer n.

Solution. Since n is odd, there exists $k \in \mathbb{Z}$ satisfying $n = 2k + 1$. If we substitute n by $2k + 1$ in the original equation, we get

$$\begin{aligned}
n^4 + 4n^2 + 11 &= (2k+1)^4 + 4(2k+1)^2 + 11 \\
&= (4k^2 + 4k + 1)^2 + 4(4k^2 + 4k + 1) + 11 \\
&= \left[4(k^2 + k) + 1\right]^2 + 4(4k^2 + 4k + 1) + 11 \\
&= \left[16(k^2+k)^2 + 8(k^2+k) + 1\right] + \left[16(k^2+k) + 4\right] + 11 \\
&= 16\left[(k^2+k)^2 + (k^2+k)\right] + 8k(k+1) + 16 \\
&\equiv 8k(k+1) \pmod{16}
\end{aligned}$$

Since either k or $k+1$ must be even (they are consecutive), we further know that $k(k+1)$ is even. Therefore $8k(k+1)$ is a multiple of 16. Hence, when n is odd, $n^4 + 4n^2 + 11 \equiv 0 \pmod{16}$, i.e., it is divisible by 16.

Problem 2. Find all points with integer coordinates lying on the line through $(\frac{1}{14}, \frac{1}{2})$ and $(1, -\frac{4}{9})$.

Solution. We wiill first compute the equation of the line through these two points:

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\frac{4}{9} - \frac{1}{2}}{1 - \frac{1}{14}} = -\frac{17}{18} \cdot \frac{14}{13} = -\frac{119}{117}$$

from which we find the point-slope form

$$y - \frac{1}{2} = -\frac{119}{117}\left(x - \frac{1}{14}\right).$$

Getting rid of all the fractions we have the standard form

$$119x + 117y = 67 \text{ (this looks a LOT nicer!)}.$$

The question is now equivalent to finding all integer pairs (x, y) satisfying the equation above. Since $119, 117, 67$ are pairwise coprime, there really isn't any shortcut. We will need to first solve the equation

$$119x + 117y = \gcd(119, 117) = 1.$$

Time to resort to Euclid's Algorithm:

$$\begin{aligned} 119 &= 1\cdot 117 + 2 && \implies 27 = (1)(119) + (-1)(117) \\ 117 &= 59\cdot 2 + 1 && \implies 1 = (1)(117) + (-58)(2) \\ &&& \quad = (1)(117) + (-58)(119) + (58)(117) \\ &&& \quad = (-58)(119) + (59)(117). \end{aligned}$$

Now we know $\begin{bmatrix} x & y \end{bmatrix}^T = \begin{bmatrix} -58 & 59 \end{bmatrix}^T$ is a particular solution to the equation $119x + 117y = 1$. If we multiply both sides of this equation by 67, we get

$$119(-58\cdot 67) + 117(59\cdot 67) = 67. \tag{1}$$

Since $-58\cdot 67$ and $59\cdot 67$ look disgusting, we will simply them a bit. Note that

$$119\cdot 117n - 117\cdot 119n = 0, \tag{2}$$

which means we can add any number of (2) to (1) without altering the 67 on the RHS. Since $\left\lfloor \frac{58\cdot 67}{117} \right\rfloor = \left\lfloor \frac{59\cdot 67}{119} \right\rfloor = 33$, we'll set $n = 33$. Then (1)+33·(2) becomes

$$119(-58\cdot 67 + 117\cdot 33) + 117(59\cdot 67 - 119\cdot 33) = 67 \implies 119(-25) + 117(26) = 67.$$

Now we've found a good-locking particular solution: $\begin{bmatrix} x & y \end{bmatrix}^T = \begin{bmatrix} -25 & 26 \end{bmatrix}^T$. The general solution can be computed by adding any multiple of $\begin{bmatrix} 117 & 119 \end{bmatrix}^T$ (which comes from equation (2)). Therefore all the lattice points on the line have coordinates of the form $(117n - 25, -119n + 26)$, $n \in \mathbb{Z}$.

Problem 3. Let nonzero $a, b \in \mathbb{Z}$. Further assume a is odd and b is even. Let $d = \gcd(a, b)$ and $e = \gcd(a+b, a-b)$. Show that $d = e$.

Solution. To prove $d = e$, it suffices to show $d \mid e$ and $e \mid d$.

The first one is obvious in the sense that $d = \gcd(a, b) \implies d \mid a$ and $d \mid b$. It follows that $d \mid a + b$ and $d \mid a - b$. Therefore d is a common divisor of $a + b$ and $a - b$ and $d \mid e$.

Now we want to show $e \mid d$. Since $e = \gcd(a+b, a-b)$, we know it divides both. Then $e \mid (a+b) + (a-b) = 2a \implies e \mid 2a$. Likewise $e \mid (a+b) - (a-b) = 2b \implies e \mid 2b$. Since the odd number a cannot have even divisors, e is odd and thus $e \mid 2a \implies e \mid a$ (or by Euclid's lemma). Likewise $e \mid b$. We've just shown that e is a common divisor of a and b. Hence $e \mid d$.

Having shown both $e \mid d$ and $d \mid e$, we conclude that indeed $d = e$.