MATH 410 Problem Set 2

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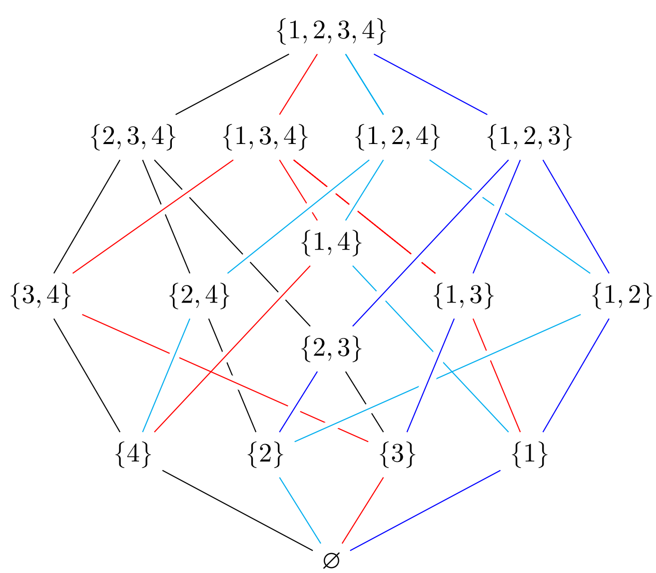
Section 1.7

1.7.5 Done in HW1.

1.7.6 Draw the poset diagram for the set of all subsets of $\{1,2,3,4\}$ under the relation $\subset$.

Solution



Section 1.8

1.8.1 State whether each of the following equations is true or false and explain.

(1) $f(A \cup B) = f(A) \cup f(B)$.

Solution

True. $\implies$ shows $\subset$ and $\impliedby$ shows $\supset$.

$$\begin{aligned} y \in f(A \cup B) &\iff y = f(x) \text{ for some } x \in A \cup B \\ &\iff y = f(x) \text{ for some } x \in A \\ &\quad\ \text{ or } y = f(x) \text{ for some } x \in B \\ &\iff y \in f(A) \text{ or } y \in f(B) \\ &\iff y \in f(A) \cup f(B). \end{aligned}$$

(2) $f(A \cap B) = f(A) \cap f(B)$. True. Proof is analogous — simply replace $\cup$ and "or" above by $\cap$ and "and".

1.8.18 Which of the following are binary operations on $\mathbb{Z}$?

(1) $a \circ b := a/b$. Nope because a/b may not be an integer while both a and b are.

(2) $a \circ b := a^2b^2$. Yes. $\circ$ is well defined and $a^2b^2 \in \mathbb{Z}$ whenever $a, b \in \mathbb{Z}$.

(3) $a \circ b = \sqrt{ab}$. Nope; same as (1).

Section 2.1

2.1.7 Show that the identity element of a group is unique. Then show that, for $a \in G$, the element a^{-1} is unique.

Proof. Let $e, e' \in G$ be identities. Treating e as the identity and using it on e' we have $ee' = e'e = e'$. On the other hand, treating e' as the identity and using it on e we have $ee' = e'e = e$. Therefore $e = e'$.

Now pick $a \in G$ and suppose a^{-1}, $a^{-1'}$ are inverses of a. Then $aa^{-1} = a^{-1}a = e$ and $aa^{-1'} = a^{-1'}a = e$. Since

$$a^{-1} = ea^{-1} = (a^{-1'}a)a^{-1} = a^{-1'}(aa^{-1}) = a^{-1'}$$

we see that inverses are indeed unique. □

2.1.10 Show that in a group G, if $a, b \in G$ and $(ab)^2 = a^2b^2$ then $ab = ba$.

Proof. Multiplying both sides by a^{-1} on the left and b^{-1} on the right, we have

$$a^{-1}(abab)b^{-1} = a^{-1}(aabb)b^{-1} \implies (a^{-1}a)ba(bb^{-1}) = (a^{-1}a)ab(bb^{-1}) \implies ba = ab.$$

□

2.1.13 Show that $|S_n| = n!$ and $|D_n| = 2n$.

Proof. For S_n, notice that, for a permutation of $\{1, 2, \ldots, n\}$, we have n options to choose for what 1 gets mapped to. Then we have $n-1$ options to choose what 2 gets mapped to. So on and so forth, until for n we only have one remaining spot to assign. Therefore we have $n(n-1)\ldots(n-(n-1))1 = n!$ options.

For D_n, notice that each "phase" of a regular n-gon is uniquely determined by the combination of its orientation and the relative position of one vertex. We pick two adjacent vertices d_1 and d_2. Once their positions are determined, so is the entire n-gon. Fixing d_1 we have two options for d_2. Staying with the same orientation [?] (i.e., clockwise or counterclockwise) we both have n possible options for d_1 and 2 as the n-gon has n vertices. Hence $|D_n| = 2n$. □

Section 2.2

2.2.4 Prove that every multiplication for a finite group is a Latin square.

Proof. Suppose c, d are in the same row of a multiplication table such that $c = d$. Assume they are on row a and that $c = ax$, $d = ay$. Multiplying both by a^{-1} on the left, we have $x = y$, i.e., c and d are in the same entry.

Likewise, suppose e, f are both in the column corresponding to b. Then for some z, w we have $zb = e$ and $wb = f$. Multiplying both by b^{-1} on the right we obtain $z = w$, so e and f must be in the same entry as well. □

2.2.7 Consider the set SO(2) consisting of matrices

$$m(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{bmatrix}$$

where $\theta \in \mathbb{R}$. Show that this is a group under matrix multiplication. What is the effect of $m(\theta)$ acting on $v = [1 \quad 0]^T$? Is this group commutative?

Solution

Starting from the second question — $m(\theta)$ rotates the vector v by θ counterclockwise. To see that these rotation matrices form a group: closure is guaranteed by the fact that compositions of rotations are still rotations; associativity is trivial as rotating by $\theta_1 + (\theta_2 + \theta_3)$ is the same as rotating by $(\theta_1 + \theta_2) + \theta_3$; identity is $m(0) = I_{2\times 2}$, i.e., not rotating at all; and inverse of $m(\theta)$ is $m(-\theta)$, i.e., rotating backwards. Indeed this group is commutative as $m(\theta_1)m(\theta_2)$ means rotating (counterclockwise) by θ_2 followed by rotating by θ_1, whereas $m(\theta_2)m(\theta_1)$ means the other way around. Both of them produces the same outcome of rotating by $\theta_1 + \theta_2$.

2.2.8 Suppose that G is a group with identity e. Show that if $g^2 = e$ for all $g \in G$ then G is Abelian.

Proof. Since $g^2 = e$ for all $g \in G$ we know $g = g^{-1}$. Take $a, b \in G$. It follows that $(ab)^2 = abab = e$. On the other hand, so is $aabb = a^2b^2$. Then the claim follows from Ex.2.1.10 by applying a^{-1} to the left and b^{-1} to the right on the equation $(ab)^2 = a^2b^2$. □