MATH 410 HW4

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2.4.12 Suppose our center's first row is of form $[a, b]$. Let $A = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$ be any element of the affine group. Then

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ax & ay+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ax & bx+y \\ 0 & 0 \end{bmatrix},$$

from which we obtain the necessary and sufficient condition $ay + b = bx + y$, independent of choice of (x, y). Letting $x = y = 1$, we have $a + b = b + 1$ and so $a = 1$. Now the original equation reduces to $y + b = bx + y$ and so $b = bx$ for all b. Now take $x = 2$. Then $b = 2b$ and so $b \equiv 2b \pmod p \implies b \equiv 0 \pmod p$ and so $b = 0$. Hence the affine group has a trivial center.

2.5.3 $o(a^k)$ denotes the smallest positive integer p such that $(a^k)^p$ becomes the identity. Note that the identity here is of form $(a^n)^m$ for some positive integer m, i.e., powers of itself. Therefore, for some $m \in \mathbb{N}$ we have

$$(a^k)^p = a^{kp} = (a^n)^m = a^{mn}.$$

Therefore this question reduces to finding the smallest p such that $n \mid kp$. Therefore for any common divisor d of n and k, one has $(n/d) \mid (kp)/d$. Letting $d := \gcd(n, k)$ gives

$$\frac{n}{\gcd(n,k)} \,\Bigg|\, \frac{k}{\gcd(n,k)}p.$$

Note that $n/\gcd(n, k)$ is coprime to $k/\gcd(n, k)$ (otherwise we can divide both sides by some nontrivial factor, and $\gcd(n, k)$ times this factor gives us an even larger gcd, contradiction). Therefore by Euclid's lemma

$$\frac{n}{\gcd(n,k)} \,\Bigg|\, p$$

and so the smallest p is $n/\gcd(n, k)$ itself. Indeed,

$$(a^k)^{n/\gcd(n,k)} = a^{kn/\gcd(n,k)} = (a^n)^{k/\gcd(n,k)} = e^{k/\gcd(n,k)} = e.$$

3.1.10 $\tau = (1576)(24)$.

$$\tau^{-1} = \begin{pmatrix} 5 & 4 & 3 & 2 & 7 & 1 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} = (6751)(42) = (1576)^{-1}(24)^{-1}.$$

The order of this permutation is $\mathrm{lcm}(3, 4) = 12$.