MATH 410 Homework 6

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March 18, 2021

3.4.1 First we enumerate the elements of $D_4 : \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$. One normal subgroup is immediate: $\{e, r, r^2, r^3\}$ (*this is normal because it has index* 2).

Before looking for other normal subgroups, first recall that D_4 is represented by $\langle r, f \mid r^4 = f^2 = e, frf = r^{-1} = r^3 \rangle$. It follows that we immediately have the following properties:

$$frf = r^3 = r^3 f^2 \implies fr = r^3 f \text{ and } fr^2 = (fr)r = r^3(fr) = r^6 f = r^2 f.$$

Now we find other normal subgroups of order 4. Suppose $H \triangleleft D_4$. Further assume $f \in H$. Then

$$r^{-1} fr = r^3 fr = r^6 f = r^2 f = fr^2 \in H$$

and so $f \in H \iff fr^2 \in H$. Notice that this also implies $r^2 \in H$. Time to check that $\{e, r^2, f, fr^2\}$ is a subgroup; we use the two-step method.

(1) Closure: obvious as the product of any two elements of H still has even power of r.

(2) Inverse: $e^{-1} = e, (r^2)^{-1} = r^2, f^{-1} = f$, and $(fr^2)^{-1} = (r^2)^{-1} f^{-1} = r^2 f = fr^2$.

Indeed $\{e, r^2, f, fr^2\}$ form a subgroup of H and since $[G : H] = 2$ it is normal.

Now, instead of assuming $f \in H$ we assume $fr \in H$. Conjugating by r gives

$$r^{-1}(fr)r = r^3 fr^2 = rf = fr^3$$

and so $fr \in H \iff fr^3 \in H$. This then automatically leads to $r^2 \in H$ as well. Now we check whether $\{e, r^2, fr, fr^3\}$ is a subgroup:

(1) Closure: trivial when involving e, so it suffices to check the remaining 9 terms: $(f^2)^2 = e, r^2(fr) = fr^3, r^2(fr^3) = fr, (fr)r^2 = fr^3, (fr)^2 = e, (fr)(fr^3) = r^2, (fr^3)(r^2) = fr, (fr^3)(fr) = r^2$, and $(fr^3)^2 = e$.

(2) Inverse: trivial for e and r^2. $(fr)^{-1} = r^{-1} f^{-1} = r^3 f = fr$ and $(fr^3)^{-1} = (r^3)^{-1} f^{-1} = rf = fr^3$.

Indeed, $\{e, r^2, fr, fr^3\}$ form a subgroup of H and since $[G : H] = 2$ it is normal.

Finally, time for normal subgroups of order 2, of which there is only one: $\{e, r^2\}$. Indeed, r^2 commutes with any element in D_4 so $g^{-1}(r^2)g = r^2(g^{-1}g) = r^2$. Notice that the subgroup $\{e, f\}$ is not normal: $r^{-1} fr = fr^2$.

3.4.5 First notice that $D_4/\left\langle R^2\right\rangle$ contains 4 elements, namely $\{e,r^2\} = \left\langle r^2\right\rangle, \{r,r^3\} = r\left\langle r^2\right\rangle, \{f,fr^2\} = f\left\langle r^2\right\rangle$, and $\{fr,fr^3\} = fr\left\langle r^2\right\rangle$. We have $\left\langle r^2\right\rangle = r^2\left\langle r^2\right\rangle, r\left\langle r^2\right\rangle = r^3\left\langle r^2\right\rangle, f\left\langle r^2\right\rangle = fr^2\left\langle r^2\right\rangle$, and $fr\left\langle r^2\right\rangle = fr^3\left\langle r^2\right\rangle$. Therefore, one way to produce the multiplication table for $D_4/\left\langle r^2\right\rangle$ is as follows:

$\cdot$	$\langle r^2\rangle$	$r\langle r^2\rangle$	$f\langle r^2\rangle$	$fr\langle r^2\rangle$
$\langle r^2\rangle$	$\langle r^2\rangle$	$r\langle r^2\rangle$	$f\langle r^2\rangle$	$fr\langle r^2\rangle$
$r\langle r^2\rangle$	$r\langle r^2\rangle$	$\langle r^2\rangle$	$fr\langle r^2\rangle$	$f\langle r^2\rangle$
$f\langle r^2\rangle$	$f\langle r^2\rangle$	$fr\langle r^2\rangle$	$\langle r^2\rangle$	$r\langle r^2\rangle$
$fr\langle r^2\rangle$	$fr\langle r^2\rangle$	$f\langle r^2\rangle$	$r\langle r^2\rangle$	$\langle r^2\rangle$

3.4.8 Let H_1, H_2 be normal subgroups of G. Clearly this is a group: the identity is in both H_1 and H_2; if $h \in H_1 \cap H_2$ then $h^{-1} \in H_1$ and $h^{-1} \in H_2$, i.e., $h^{-1} \in H_1 \cap H_2$. Closure is also clear as $h_1, h_2 \in H_1 \cap H_2 \implies h_1h_2 \in H_1$ and $h_1h_2 \in H_2$ and thus $h_1h_2 \in H_1 \cap H_2$.

To see this intersection is normal, pick $h \in H_1 \cap H_2$. Also pick arbitrary $g \in G$. Since H_1 is normal, $g^{-1}hg \in H_1$; likewise, $g^{-1}hg \in H_2$. Therefore $g^{-1}hg \in H_1 \cap H_2$. □

3.4.9 For $\implies$, assume G/H is cyclic. Pick any $e \neq g \in G$. It follows that $gH \in G/H$ and gH is not the identity of G/H (since H is). By assumption it generates G/H, so every $\tilde{g}H \in G/H$ can be written as $(gH)^k$, but

$$(gH)^k = g^kH^k = g^kH \implies \tilde{g} = g^k.$$

In other words, every $\tilde{g} \in G$ can be expressed as a power of g, i.e., $G = \langle g\rangle$.

For $\impliedby$, assume G is cyclic. Let g be a generator. Pick any $\tilde{g}H \in G/H$. It follows that $\tilde{g} = g^k$ for some k. Then,

$$\tilde{g}H = g^kH = g^kH^k = (gH)^k.$$

Therefore gH generates G/H, and we are done. □

3.4.11 If G is cyclic then we are immediately done since $G = \langle g\rangle \implies o(g^5) = 3$.

Actually, it doesn't matter. By Lagrange's theorem, any $g \in G$ can have order $1, 3, 5, 15$. Suppose for contradiction that no $g \in G$ has order 3. It follows that no $g \in G$ can have order of multiples of 3, i.e., 15, either. Clearly if $g \neq e$ then $o(g) \neq 1$ so the only remaining possibility is if all 14 non-identity $g \in G$ have order 5. Pick $g_1 \in G$ and consider $\langle g_1\rangle$. Clearly there exist $g_2 \in G \setminus \langle g_1\rangle$. Now we consider $\langle g_2\rangle$. Notice that the intersection of two subgroups is also a subgroup (easy to check closure, identity, inverse). Of course this is a proper subgroup, and since 5 is prime, $|\langle g_1\rangle \cap \langle g_2\rangle| = 1$, i.e., the intersection is simply $\{e\}$. This tells us $|\langle g_1\rangle \cup \langle g_2\rangle| = 9$, and thus we have 5 unused elements from G. Take such g_3 and consider $\langle g_3\rangle$. It becomes clear that the pairwise intersection of $\langle g_1\rangle, \langle g_2\rangle$, and $\langle g_3\rangle$ is $\{3\}$, so inclusion-exclusion gives

$$|\langle g_1\rangle \cup \langle g_2\rangle \cup \langle g_3\rangle| = 13.$$

Now there exists just one unused element in G that has order 5. But we do not have enough room for another cyclic group of order 5. Contradiction! Thus at least one element of G has order 3.

Maybe it does matter — it is also possible to prove the much stronger claim that if $|G| = 15$ *then* G *is cyclic, in which case the claim follows from the first sentence.* □

3.4.14 If $gH \in G/H$ has order n, then

$$(gH)^n = g^nH = eH \implies g^n = e \implies o(g) = e. \qquad \square$$

3.5.1 Indeed, $\phi(x+y) = \phi(x)\phi(y)$ as $e^{x+y} = e^xe^y$. This is indeed an isomorphism as shown in the previous HW.

3.5.5 We have shown that $\mathrm{GL}(n,\mathbb{R})$ is a group in class already: closure is guaranteed as $\det(AB) = \det(A)\det(B)$; identity is simply $I_{2\times 2}$, and inverse is just the inverse of a matrix. To show that the determinant operator defines a homomorphism, we again use the fact that $\det(AB) = \det(A)\det(B)$. The claim follows. $\square$

3.5.8 (1) No, this is not: $(1+1)^3 = 8 \neq 1^3 + 1^3$.

(2) Yes. Notice that f actually has the same effect as the identity map: $f([0]) = [0], f([1]) = [1]$, and $f([2]) = [8] = [2]$. Therefore it follows naturally that $f(ab) = ab = f(a)f(b)$.

3.5.13 **Second Isomorphism Theorem**. We first show that the notions of $K/(N\cap K)$ and KN/N are well-defined.

(1) $K/(N\cap K)$: it suffices to show that $N \cap K$ is a normal subgroup of K. The intersection of two groups is clearly a group. To show it is normal, pick any $x \in N \cap K$ and $k \in K$. Then

$$x \in N \implies k^{-1}xk \in N \text{ and } x, k \in K \implies k^{-1}xk \in K.$$

Therefore $k^{-1}(N \cap K)k \subset N \cap K$ for all $k \in K$. To show the other inclusion, simply notice that

$$k^{-1}(N\cap K)k \subset N \cap K \implies k\left[k^{-1}(N\cap K)k\right]k^{-1} = (N\cap K) \subset k(N\cap K)k^{-1}. \qquad (\Delta)$$

This result is enough because k is chosen arbitrarily; in other words, for any $\tilde{k} \in K$, we always have

$$(N\cap K) \subset \tilde{k}^{-1}(N\cap K)\tilde{k}$$

if we simply apply the result from (Δ), which holds for all elements of G, including $\tilde{k}^{-1}$.

(2) KN/N: we need to first show that KN is a subgroup of G and then $N \triangleleft KN$. Indeed, KN has closure because, for $k_1n_1, k_2n_2 \in KN$, we have

$$(k_1n_1)(k_2n_2) = k_1(k_2\tilde{n}_1)n_2 = (k_1k_2)(\tilde{n}_1n_2) \in KN$$

where the existence of $\tilde{n}_1$ satisfying $n_1k_2 = k_2\tilde{n}_1$ is guaranteed since N is normal. Identity is simply $ee = e$ and $(kn)^{-1} = n^{-1}k^{-1} = k^{-1}\hat{n}^{-1} \in KN$, where the existence of $\hat{n}^{-1}$ is once again guaranteed since N is normal.

$N \triangleleft KN$ is trivial as $N \triangleleft G$ and $KN \subset G$.

Proof of the S.I.T. By the hint, consider $T : K \to KN/N$ by $k \mapsto kN$. Indeed this is a homomorphism:

$$T(k_1k_2) = k_1k_2N = k_1k_2N^2 = k_1Nk_2N = T(k_1)T(k_2).$$

Suppose $x \in \ker(T) \subset N$. Then $xN = N$ so $x \in N$. Therefore $x \in N \cap K$. By the F.I.T., we indeed have the desired isomorphism relation

$$K/\ker(T) = K/(N\cap K) \cong (KN)/N. \qquad \square$$

End of HW6