MATH 410 Homework 8

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3.7.18 Find the number of tiny bracelets of four beads that can be made with two colors of beads.

Solution. Like the example shown in class, here we consider how D_4 acts on the bracelets — in particular, we say that two color patterns are essentially the same if one can be obtained by the other via reflection or rotation. Since there are 4 beads, these actions naturally identity with group actions under D_4. The number of different necklaces, therefore, is the number of orbits under such action. By Burnside's lemma,

$$\#\text{Orbits} = \frac{1}{|D_4|} \sum_{\sigma \in D_4} |\text{Fix}(\sigma)|.$$

Now we just need to find the size of $\text{Fix}(\sigma)$ for each $\sigma \in D_4$.

Action	Corresponding σ	Number of σ's	$\lvert\text{Fix}(\sigma)\rvert$	Total
Reflection across diagonal	fr, fr^3	2	$2^3 = 8$	16
Reflection over sides	f, fr^2	2	$2^2 = 4$	8
Rotation of 90°	r, r^3	2	2	4
Rotation of $180^{(\circ)}$	r^2	1	$2^2 = 4$	4
Nothing	e	1	$2^4 = 16$	16

It follows that

$$\#\text{Orbits} = \frac{16 + 8 + 4 + 4 + 16}{8} = 6,$$

i.e., there are six *different* bracelets consisting of four beads, each with two color options.

3.7.19 In how many ways can we paint a square floor made of of nine square tiles using purple and orange paint?

Solution. Similar to above, but here we have R_4 (the rotation group) acting on the floor, and we easily obtain the following diagram:

Action	Corresponding σ	Number of σ's	$\lvert\text{Fix}(\sigma)\rvert$	Total
Rotation by 90°	r, r^3	2	$2^3 = 8$	16
Rotation by 180°	r^2	1	$2^5 = 32$	32
Identity	e	1	$2^9 = 512$	512

It follows that the total number of ways is $(16 + 32 + 512)/4 = 140$.

4.7.20 In how many ways can we color a cube's faces with four colors?

Solution. The key is to notice that the symmetry group of a cube is isomorphic to S_4. To see this, we name the vertices of the top face of a cube $1-2-3-4$ and also name the other four $1-2-3-4$, each corresponding to the opposite of its counterpart (i.e., if we draw a line between the two 1's, it should go through the center. Then we can identify all actions on the cube by the following:

Action	Corresponding σ	Number of σ's	$\lvert\mathrm{Fix}(\sigma)\rvert$	Total
Identity	e	1	$2^6 = 64$	64
Edge-midpoint rotation	(ab)	6	$2^3 = 8$	48
Face-midpoint rotation	$(abcd)$	6	$2^3 = 8$	48
Face-midpoint rotation, twice	$(ab)(cd)$	3	$2^4 = 16$	48
Diagonal	(abc)	8	$2^2 = 4$	32

Therefore the total number of ways to paint a cube using two colors is

$$\frac{64+48+48+48+32}{24} = 10.$$

5.2.5 (1) $2\mathbb{Z}\cup 5\mathbb{Z}$ is not a subring of $\mathbb{Z}$ — it is not even closed under addition: $2+5=7\notin 2\mathbb{Z}\cup 5\mathbb{Z}$.

(2) $2\mathbb{Z}+5\mathbb{Z}=\mathbb{Z}$ because $2,5$ are co-prime and Bézout's identity guarantees that any integer can be represented by a $\mathbb{Z}$-combination of 2 and 5.

(3) $2\mathbb{Z}\cap 5\mathbb{Z}=10\mathbb{Z}$ because a number is both a multiple of 2 and of 5 if and only if it is a multiple of 10.

5.2.9 Just like in $\mathbb{Z}[x]$, all units must have degree 0 because it must not exceed that of 1, a degree 0 polynomial. What's different is that any nonzero constant coefficient is a unit in $\mathbb{R}[x]$ because any nonzero number in $\mathbb{R}$ has a multiplicative inverse.

5.3.3 (1) It is an integral domain because two nonzero polynomials can never multiply to get zero, but it is not a field because x^2 has no multiplicative inverse (namely $1/x^2$ is not a polynomial), for example.

(2) This is not an integral domain: consider, for example, $f := \chi_{[0,1]}$ and $g := \chi_{[2,3]}$. Their product is zero pointwise but clearly neither is the zero function.

5.3.9 (1) This is an integral and also a field. Every nonzero $a+bi$ can be written as $re^{i\theta}$ so it always has an inverse $r^{-1}e^{-i\theta}$. (It follows that if $r_1e^{i\theta 1}r_2e^{i\theta 2}=0$ then $r_1r_2=0$, i.e., either $r_1e^{i\theta 1}=0$ or $r_2e^{i\theta 2}=0$.)

(2) Not an integral domain: $[2][6]=[12]=[0]$.

(3) Not an integral domain:

$$\begin{bmatrix}[1] & [1]\\ [1] & [1]\end{bmatrix}\begin{bmatrix}[1] & [1]\\ [1] & [1]\end{bmatrix} = \begin{bmatrix}[2] & [2]\\ [2] & [2]\end{bmatrix} = 0.$$

(4) Assuming the textbook has made a typo (i.e., mod 12 not mod 11), $\mathbb{Z}/11\mathbb{Z}$ is indeed an integral domain and a field, as 11 is prime and everything besides 0 in $\mathbb{Z}/11\mathbb{Z}$ has a multiplicative inverse.

(5) Not an integral domain: $(0,1)\cdot(1,0)=(0,0)$.

(6) Both an integral domain and a field; clear enough.

(7) Not necessarily an integral domain. (It is an integral domain iff R itself is.) For nonexample, consider $R := \mathbb{Z}/6\mathbb{Z}$ where $(2x)\cdot(3x) = 0$.

5.3.10 (1) Not an integral domain. Consider f whose graph connects $(0,1),(1/2,0)$, and $(1,0)$ (points in $\mathbb{R}^2$) and g whose graph connects $(0,0),(1/2,0)$, and $(1,0)$. It follows that, for every $x \in [0,1]$, either $f(x) = 0$ or $g(x) = 0$ so $fg \equiv 0$, but it is clear that neither of them is the zero function.

To <u>not</u> be a unit, f has to be zero at some $x \in [0,1]$. To <u>not</u> be a zero divisor, the level set of f at level 0 must be totally disconnected for example $f(x) = 1/2 - x$. If $fg \equiv 0$ then clearly $g \equiv 0$ on $[0,1] \setminus \{0.5\}$ but by continuity we also have $g(0.5) = 0$, i.e., $g \equiv 0$.

(2) This is also not an integral domain. Consider $f, g \in C(\mathbb{Z}/2\mathbb{Z})$ defined by

$$f(x) = \begin{cases} 1 & x = [0] \\ -1 & x = [1] \end{cases} \qquad g(x) = 1 \text{ for all } x \in \mathbb{Z}/2\mathbb{Z}.$$

It follows that

$$(f * g)([0]) = f([0])g([0]) + f([1])g([1]) = 1 + -1 = 0$$

and

$$(f * g)([1]) = f([1])(g[0]) + f([0])g([1]) = -1 + 1 = 0.$$

However it is clear that $f, g \not\equiv 0$. Functions like $f([0]) = 0, f([1]) = 1$ is neither a unit nor a zero divisor.

5.3.11 (a) (1) None.

(2) $[2],[3],[4],[6]$.

(3) Checking by brute force suggests that the only zero divisors are the "one matrix" above and four more, each with one $[1]$ and three $[0]$'s.

(4) None.

(5) $(n,0)$ and $(0,m)$ where $n,m \in \mathbb{Z}$.

(6) None.

(7) Clearly all the polynomials of x with zero divisors of R as coefficients are zero divisors of $R[x]$, but I am not sure if there are other zero divisors of $R[x]$. After all R is just an arbitrary ring.

(b) (1) Every nonzero complex number.

(2) Every nonzero element besides $[2],[3],[4]$, and $[6]$.

(3) Every other nonzero matrix in the ring.

(4) Every nonzero element.

(5) Everything (x,y) with both x,y nonzero.

(6) Every nonzero rational number.

(7) Hmmm. Every polynomial with *at least* one coefficient not being a zero divisor of R??

(c) At least for (1) to (6), the relation is that they together constitutes all nonzero elements of R.

5.4.4 Notice that we have $[x^2-2] = [0]$, i.e., $[x]^2 - [2] = [0]$. Therefore anything in $\mathbb{Q}[x]/\langle x^2-2\rangle$ can be idenfitied with a something whose power of x does not exceed 1. Heuristically this means $[x]$ behaves like $\sqrt{2}$ and we identify $\mathbb{Q}[x]/\langle x^2-2\rangle$ with $\mathbb{Q}[\sqrt{2}]$.

5.4.5 Since A is an ideal, for any $s \in R$, $us \in A$. Oh the other hand, rxs is clearly a product of r times something in R, so this set is at least a one-sided ideal. For the other side, we need to use the fact that R is commutative and obtain the result that $srx = r(sx)$. This would show that the set is a two-sided ideal, as claimed.