CS 532 Homework 4

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Solution to problem 1. Recall that in lecture, we discussed two randomized rounding scheme. One sets $\mathbb{P}(x_i = 1) = 1/2$, whereas the other uses $\mathbb{P}(x_i = 1) = y_i^*$ where y_i^* is the optimal solution to the LP relaxation of MAX-SAT. We showed that the first one has a probability of $1 - 2^{-\ell_j}$ of satisfying a clause c_j with length ℓ_j, and the second has $1 - (1 - 1/\ell_j)^{\ell_j}$. If we simply randomly pick one of the two algorithms, then expected probability of c_j being satisfied is the average between $1 - 2^{-\ell_j}$ and $1 - (1 - 1/\ell_j)^{\ell_j}$, which we have claimed in lecture $\geqslant 3/4$.
This completes the proof, since randomly picking precisely yields $\mathbb{P}(x_i = 1) = 1/2 \cdot 1/2 + 1/2 \cdot y_i^* = y_i^*/2 + 1/4$.

Solution to problem 2. (1) When $x_i \in \{-1, 1\}$ and we let $x_i = 1$ to represent assignment of TRUE, then the clause $(x_i \vee x_j)$ is true iff $1 - x_i$ or $1 - x_j = 0$, i.e., if $(1 - x_i)(1 - x_j) = 0$. Thus, assuming $x_i \in \{-1, 1\}$,

$$1 - (1 - x_i)(1 - x_j)/4 = \mathbf{1}[x_i \vee x_j].$$

Following the hint, to invoke vector relaxation we represent each clause $(x_i \vee x_j)$ by

$$1 - (v_0 - v_i)(v_0 - v_j)/4, \qquad v_0, v_i, v_j \in \mathbb{R}^n \text{ with unit norm.}$$

Expanding gives

$$1 - (v_0 - v_i)(v_0 - v_j)/4 = 1 - \frac{\|v_0\| - v_0^T v_i - v_0^T v_j + v_i^T v_j}{4} = \frac{1 + v_0^T v_i}{4} + \frac{1 + v_0^T v_j}{4} + \frac{1 - v_i^T v_j}{4}. \qquad (*)$$

Therefore, the "integer quadratic program" can be phrased as $\sum_{\text{clauses } C_k} w_k c(C_k)$ where if $C_k = (x_i \vee x_j)$ then $c(C_k)$ is defined as in (*), subject to $v_0, v_i \in \mathbb{R}^n$, with norm 1.

(2) Let $\theta_{i,j}$ be the angle between v_i and v_j. We perform a rounding similar to the one in MAX-CUT: pick a random vector and correspondingly two hemispheres of the unit sphere, then round all variables in one hemisphere to $+1$ and all in the other to -1. Therefore, $1 + v_0^T v_i$ is either 2 or 0, where the former has probability $\mathbb{P}(1 + v_0^T v_i) = \mathbb{P}(v_0, v_i \text{ in same hemisphere}) = 1 - \theta_{0,i}/\pi$. Likewise, $\mathbb{P}(1 - v_i^T v_j = 2) = \theta_{i,j}/\pi$.
Because $\cos\theta$ is an even function, $f(\theta) = \dfrac{2(\pi - \theta)}{\pi(1 + \cos\theta)}$ attains the same minimum as $\dfrac{2\theta}{\pi(1 - \cos\theta)}$, which is ≈ 0.878. Applying this result term-by-term to the three terms in (*), we see that overall this rounding scheme also gives an 0.878-approximation of MAX-2SAT.

Solution to problem 3. (1) Essentially, $z_{i,j}$ can be viewed as the indicator that $(i \to j)$ is an edge leaving U. In the integer constraint $x_i \in \{0, 1\}$, we indicate whether node i is in U. Therefore, in other words, in order for $(i \to j)$ to leave U there are two necessary conditions: (i) $i \in U$, meaning $x_i = 1$, and (ii) $j \notin U$, meaning $x_j = 0$. With the assumption that $0 \leqslant z_{i,j} \leqslant 1$, this translates to the two constraints $z_{i,j} \leqslant x_i \leqslant 1 - x_j$. And of course the objective measures the total weight of edges going from U to W.

(2) This rounding algorithm gives

$$\mathbb{P}(i \in U, j \notin U) = \left(\frac{1}{4} + \frac{x_i}{2}\right)\left(1 - \left(\frac{1}{4} + \frac{x_j}{2}\right)\right) = \left(\frac{1}{4} + \frac{x_i}{2}\right)\left(\frac{1}{4} + \frac{1 - x_j}{2}\right).$$

The constraints force $z_{i,j} \leqslant \min(x_i, 1 - x_j)$. If $x_i \leqslant 1 - x_j$ then

$$\mathbb{P}(i \in U, j \notin U) \geqslant \left(\frac{1}{4} + \frac{x_i}{2}\right)^2 \geqslant \frac{x_i}{2} = \frac{\min x_i, 1 - x_j}{2} \geqslant \frac{z_{i,j}}{2}$$

and likewise for the other case. This shows that the rounding scheme is $1/2$-approximate.

Solution to problem 4. (1) For each of the m samples sample $x_1, \cdots, x_m$, define an indicator random variable X_j that evaluates to 1 if $x_j \in S_i$. In hindsight X_j should follow a Bernoulli distribution with success probability $p = v_i/v_{i+1}$. Let $\hat{p}$ be the sample mean $m^{-1}\sum_{j=1}^{m} X_j$. We use the following formulation of Hoeffding's inequality: if $X_1, \cdots, X_m$ are independent and in $[a, b]$, and $S_m = \sum_i X_i$, then

$$\mathbb{P}(|S_m - \mathbb{E}S_m| \geqslant \epsilon) \leqslant 2\exp\left(-\frac{2\epsilon^2}{\sum_{i=1}^{m}(b-a)^2}\right).$$

Substituting $[a, b]$ with $[0, 1]$ and adding a factor of m to both sides, we see

$$\mathbb{P}(|\hat{p} - p| \geqslant \epsilon) \leqslant 2\exp(-2m\epsilon^2).$$

Now we just want m sufficiently large so $|q_i - v_i/v_{i+1}| \geqslant \epsilon$ with probability $\leqslant 2\exp(-2m\epsilon) \leqslant \delta$. This gives $-2m\epsilon^2 \leqslant \log(\delta/2)$, or $m \geqslant \log(2/\delta)/(2\epsilon)^2$ which is indeed $\mathcal{O}(\epsilon^{-2}\log\delta^{-1})$.

(2) Define $s_i = v_{i+1}/v_i$. Observe that we can express v_n as a nested product $v_n = v_0 \cdot \prod_{i=0}^{n-1}(v_{i+1}/v_i) = v_0 \cdot \prod_{i=0}^{n-1} s_i$, where each of these ratios s_i can be approximated fairly well in polynomial time. (This is because by (1), we can approximate the reciprocal $1/s_i$ arbitrarily well, and $x \mapsto 1/x$ is continuous on $(0, \infty)$ and $1/s_i = v_i/v_{i+1} \geqslant 0.5 > 0$.)

Before we determine the correct values for ϵ, δ, suppose that for each s_i, we can find $\hat{s}_i$ such that $\mathbb{P}(|\hat{s}_i - s_i| \geqslant \epsilon) \leqslant \delta$. By assumption, $v_i/v_{i+1} \geqslant 0.5$, and clearly $v_i \leqslant v_{i+1}$, so $s_i = v_{i+1}/v_i \in [1, 2]$. This implies that with probability $\geqslant 1 - \delta$, the relative error satisfies

$$e_i = \left|\frac{\hat{s}_i - s_i}{s_i}\right| \leqslant \left|\frac{\epsilon}{s_i}\right| \leqslant \epsilon. \tag{*}$$

Define our estimator $\hat{v}_n$ to be $v_0 \cdot \prod_{i=0}^{n-1}\hat{s}_i$. Then, if we want $0.5v_n < \hat{v}_n < 1.5v_n$, we need

$$\frac{1}{2}\cdot\prod_{i=0}^{n-1} s_i \leqslant \prod_{i=0}^{n-1}\hat{s}_i < \frac{3}{2}\cdot\prod_{i=0}^{n-1} s_i.$$

Observe that if $x < \log(3/2)/n$, then

$$(1 + x)^n < (1 + \log(3/2)/n)^n \approx \exp(\log(3/2)) = \frac{3}{2},$$

and likewise $(1 - x)^n > 2/3 > 1/2$. Therefore, $\epsilon = \log(3/2)/n$ is what we seek, for if $|\hat{s}_i - s_i| < \epsilon$ for all i, then

$$\frac{1}{2} < (1-\epsilon)^n \leqslant \underbrace{\prod_{i=0}^{n-1}\frac{\hat{s}_i}{s_i}}_{=\hat{v}_n/v_n} \leqslant (1+\epsilon)^n < 3/2.$$

What about probabilities? We want $|\hat{s}_i - s_i| < \epsilon$ for each i with probability $\geqslant 1 - 1/n$. A union bound on the complement suggests that setting $\delta = 1/n^2$ suffices.

To sum up:

Algorithm 0: Polynomial Time Estimator of v_n

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let ϵ = log(3/2)/n and δ = 1/n^2
for i = 0, ⋯, n − 1 do
    Use part (1) to find an estimator ŝ_i of v_{i+1}/v_i such that
      ℙ(|ŝ_i − v_{i+1}/v_i| ⩾ ϵ) ⩽ δ
return v̂_n = v_0 · ∏_{i=0}^{n−1} ŝ_i
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