Beginning of 08/25

Definitions and logistics...

Definition: Expander

In this course we will deal with undirected graphs with edge weights unless otherwise stated. A graph $G = (V, E)$ with normalized adjacency matrix $A = D^{-1/2}AD^{-1/2}$:

- is a **one-sided λ-expander** if the second largest eigenvalue $\lambda_2(A) \leqslant \lambda$;
- is a **two-sided λ-expander** if $\lambda_2(A), |\lambda_n(A)| \leqslant \lambda$.

Intuition: this graph is well-connected; in order to separate the graph, one needs to remove many graphs.

Examples that can be solved using spectral information

(1) Given $G = (V, E)$, draw it in $\mathbb{R}^n$ such that connected vertices are close to each other.

(2) Find the sparsest cut S in G that minimizes $|\partial S|/|S|$.

(3) A linear code C_k is a k-dimensional subspace of $\mathbb{F}^n$. Consider transmitting encoded information over a faulty channel so that extra information needs to be sent to overcome this. A few things we care about: (i) rate r (so not too much redundant information is needed), (ii) distance $\delta = \min_{x \in C_k \setminus \{0\}} \text{hammingweight}(x)$ which measures the tolerance between two strings, and (iii) decoding which measures how efficient the algorithm is at decoding *into the uncorrected, redundant string*. Question: can we constraint a family $\{C_k\}$ such that there exist r_0, δ_0, and $C > 0$ such that $r(C_k) \geqslant r_0$, $\delta(C_k) \geqslant \delta_0$, and the decoding algorithm takes at most Cn time (constant rate, constant distance, and linear decoding time).

(4) BPP (bounded probabilistic polynomial time): a language L if there exists a PPT-TM (probabilistic polytime Turing machine) M such that:

- if $x \in L$ then it accepts w.p. $\geqslant 2/3$, and
- if $x \notin L$, then it rejects w.p. $\geqslant 2/3$.

To reduce error within ϵ for a ℓ-bit string, one naïve operation is to repeat for $\ell \log \epsilon^{-1}$ times and take majority vote. Question: can we reduce this to $\ell + \mathcal{O}(\log \epsilon^{-1})$.

Basic notations

- Given $G = (V, E)$, define the diagonal matrix D, adjacency matrix A, and normalized adjacency matrix (with abuse of notation A), like before.
- Define $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues sorted in decreasing values, with eigenvectors v_i.
- Define the Laplacian $L = D - A$ and the normalized Laplacian $\overline{L} = D^{-1/2}LD^{-1/2} = I - \overline{A}$. Sort the spectrum of L in **increasing order**: $\gamma_1 \leqslant \cdots \leqslant \gamma_n$ with eigenvectors $u_1, \cdots, u_n$.

Some facts.

- $\lambda_1 = 1$ and $v_1 \propto D^{-1/2}\mathbf{1}$. In addition, $\lambda_n \geqslant -1$.
- $\gamma_i = 1 - \lambda_i$ which explains why we sort the γ's in increasing order. This further implies $\gamma_1 = 0$.
- $u_i = v_i$ — same eigenspace.
- Using the definition of λ-expanders: if G is a λ-expander, then $\gamma_2 \geqslant 1 - \lambda$.
- There is a variational perspective to define λ's: given a matrix M with increasing spectrum $\gamma_1 \leqslant \cdots \leqslant \gamma_n$, and eigenvalues are $u_1, \cdots, u_n$, then

$$\gamma_i = \min_{x \perp \mathrm{span}(u_1, \cdots, u_{i-1})} \frac{x^T M x}{\langle x, x \rangle}.$$

- We want to write $f^T L f$ and $f^T \overline{L} f$ in nicer ways:

$$f^T L f = f^T D f - f^T A f = \sum_{v \in V} f(a)^2 d(a) - \sum_{(a,b)\in E} 2 f(a) f(b) = \sum_{(a,b)\in E} (f(a) - f(b))^2.$$

  Likewise,

$$f^T \overline{L} f = \sum_{(a,b)\in E} \left(\frac{f(a)}{\sqrt{d(a)}} - \frac{f(b)}{\sqrt{d(b)}} \right)^2.$$

Conductance

Definition: Conductance

Given $G = (V, E)$ and $S \subset V$, define cut S's conductance as

$$\varphi(S) = \frac{|\partial S|}{\min(\mathrm{vol}(S), \mathrm{vol}(S^c))}$$

where ∂S is the set of edges between (E, E^c), and $\mathrm{vol}(S) = \sum_{a \in S} d(s)$, and define the conductance of the graph by $\varphi(G) = \min_{S \subset V} \varphi(S)$.

Furthermore, the minimizer S gives the sparsest cut. Cheeger's inequality gives $\gamma_2/2 \leqslant \varphi(G) \leqslant \sqrt{\gamma_2}$. Equalities can be attained by hypercubes and cycles, respectively.

Proof of $\gamma_2/2 \leqslant \varphi(G)$: let S be the set minimizing $\varphi(G)$ with $\mathrm{vol}(S) \leqslant 1/2\mathrm{vol}(V)$. Let $f = \mathbf{1}_S - |S|/n \cdot \mathbf{1}$. It follows from $(D - A)\mathbf{1} = \mathbf{0}$ that $f^T L f = \mathbf{1}_S^T L \mathbf{1}_S = |\partial S|$. On the other hand,

$$\begin{aligned} f^T D f &= \langle \mathbf{1}_S, D\mathbf{1}_S \rangle - 2 \langle \mathbf{1}_S, D|S|/n\mathbf{1} \rangle + \left(\frac{|S|}{n} \right)^2 \langle \mathbf{1}, D\mathbf{1} \rangle \\ &= \mathrm{vol}(S) - \frac{2|S|}{n} \mathrm{vol}(S) + \left(\frac{|S|}{n} \right)^2 \mathrm{vol}(V). \end{aligned}$$

Since $f^T D f = \mathrm{vol}(S)$. Recall that $u_1 = D^{1/2\mathbf{1}}$ so that

$$\begin{aligned} \gamma_2 &= \min_{x \perp D^{1/2}\mathbf{1}} \frac{x^T D^{-1/2} L D^{-1/2} x}{\langle x, x \rangle} = \min_{f \perp \mathbf{1}} \frac{f^T L f}{f^T D f} \\ &\leqslant \frac{|\partial S|}{\mathrm{vol}(S) - 2|S|\mathrm{vol}(S)/n + |S|^2/n^2 \mathrm{vol}(V)}. \end{aligned}$$

Rewrite the denominator as $\mathrm{vol}(S)[(1-a)^2+a^2]$ where $a = |S/n|$ and since the bracketed term $\geqslant 1/2$, the claim follows.