CS590.05 Expanders Homework 1

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Solution to problem 1. We prove by contradiction. Suppose there exist ϵ, F such that $0 \leqslant \epsilon < \phi(G)$ and $|F| \leqslant \epsilon |E|$, such that the subgraph $G' = (V, E \backslash F)$ breaks the claim.

Following the hint, let $C_1, \cdots, C_k$ be the connected components of G'. By the definition of conductance, if $\mathrm{vol}(C_i) \leqslant \mathrm{vol}(V)/2$ for all i, then we may write

$$\frac{|E(C_i, V \backslash C_i)|}{|\mathrm{vol}(C_i)|} \geqslant \phi(G) \qquad \Longrightarrow \qquad \sum_{i=1}^{k} |E(C_i, V \backslash C_i)| \geqslant \phi(G) \sum_{i=1}^{k} \mathrm{vol}(C_i) = \phi(G) \cdot \mathrm{vol}(V).$$

Observe that the LHS $\sum_{i=1}^{k} |E(C_i, V \backslash C_i)|$ is bounded by twice the number of edges removed (i.e. $2|F|$), so

$$\phi(G) \cdot \mathrm{vol}(V) \leqslant \sum_{i=1}^{k} |E(C_i, V \backslash C_i)| \leqslant 2|F| \leqslant 2\epsilon |E| = \epsilon \cdot \mathrm{vol}(G) < \phi(G) \cdot \mathrm{vol}(G),$$

contradiction. Hence we may WLOG assume $\mathrm{vol}(C_1) > \mathrm{vol}(V)/2$. All edges between C_1 and C_1^c must have been deleted, so

$$|E(C_1, C_1^c)| \leqslant |F| = \epsilon \cdot \mathrm{vol}(V)/2.$$

Using conductance on C_1^c yields

$$\frac{|E(C_1, C_1^c)|}{\mathrm{vol}(C_1)} \geqslant \phi(G) \qquad \Longrightarrow \qquad \phi(G) \cdot \mathrm{vol}(C_1^c) \leqslant |E(C_1, C_1^c)| \leqslant \epsilon \cdot \mathrm{vol}(V)/2.$$

Rearranging gives

$$\mathrm{vol}(C_1) = \mathrm{vol}(V) - \mathrm{vol}(C_1^c) \geqslant \left(1 - \frac{\epsilon}{2\phi(G)}\right) \cdot \mathrm{vol}(V),$$

as claimed. $\square$

Verdict (Q1): basically identical, 2pts / 2pts.

Solution to problem 2. We keep notations consistent with in-class versions. Let $\overline{A}$ be the normalized adjacency matrix and $\overline{L}$ the normalized Laplacian. Recall that [we use $d(u)$ to denote the degree of u]

$$x^T \overline{L} x = (D^{-1/2}x)^T (D - A)(D^{-1/2}x) = \frac{1}{2} \sum_{(u,v) \in E} \left(\frac{x_u}{\sqrt{d(u)}} - \frac{x_v}{\sqrt{d(v)}} \right)^2 \geqslant 0.$$

Therefore all eigenvalues of $\overline{L}$ are non-negative. Further, using $(a-b)^2 \leqslant 2(a^2 + b^2)$ we also have

$$x^T \overline{L} x \leqslant \frac{1}{2} \sum_{(u,v) \in E} 2 \left(\frac{x_u^2}{d(u)} + \frac{x_v^2}{d(v)} \right) = \sum_v x_v^2 = x^T x.$$

Hence all eigenvalues of $\overline{L}$ are $\leqslant 2$. It follows then from $\overline{A} = I - \overline{L}$ that all eigenvalues of $\overline{A}$ must be in $[1, -1]$.

Now suppose $\lambda_{\min}(\overline{A}) = -1$. Let x be a corresponding eigenvector. Writing $y = D^{-1/2}x$ we have $Ay = -Dy \Leftrightarrow$

$(D+A)y = 0$. However, on the other hand, the quadratic term

$$y^T(D+A)y = \sum_v d(v)\cdot y_v^2 + 2\sum_{(u,v)\in E} y_u y_v = \sum_{(u,v)\in E}(y_u+y_v)^2 \geqslant 0,$$

and it is clear that this term can be made $=0$ iff $y_u = -y_v$ for every edge (u,v). Therefore every edge connects a vertex where y is positive to one where y is negative, which naturally gives a bipartition of V.

Conversely, suppose G is bipartite with $V = S \cup T$. With abuse of notation, let $x \in \{\pm 1\}^V$ be $+1$ on S and -1 on T, and let $y = D^{1/2}x$. Then since all edges go across (S,T),

$$\overline{A}x = D^{-1/2}AD^{-1/2}D^{1/2}y = D^{-1/2}Ay = D^{-1/2}(-Dy) = -D^{-1/2}y = -x,$$

so -1 is an eigenvalue. □

Verdict (Q2): the proofs are mostly isomorphic, 2pts / 2pts. Small typos like $\mathrm{spectrum}(\overline{A}) \subset [1,-1]$.

Solution to problem 3. For each v write $M(v) = (x_v, y_v)$. Then, the objective is to minimize

$$\begin{aligned}\sum_{(u,v)\in E}\|M(u)-M(v)\|_2^2 &= \sum_{(u,v)\in E}(x_u-x_v)^2 + \sum_{(u,v)\in E}(y_u-y_v)^2\\ &= x^TLx + y^TLy = \mathrm{tr}(X^TLX),\end{aligned}$$

where L is the *un*normalized Laplacian ($= D - A$), $x = (x_v) \in \mathbb{R}^n$, $y = (y_v) \in \mathbb{R}^n$, and $X = [x\ \ y]$. The constraint is that $\sum_{u\in V}\|M(v)\|_2^2 = \|X\|_F^2 = 1$.
Let $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_n$ be the eigenvalues of L and let $u_1, \cdots, u_n$ be the corresponding orthonormal eigenvectors; from class we know that $\gamma_1 = 0$ and that $u_1 \propto \mathbf{1}$. Under the spectral decomposition $L = U\Lambda U^T$, writing $Y = U^TX$, the objective and the constraints become

$$\mathrm{tr}(X^TLX) = \mathrm{tr}(Y^T\Lambda Y) = \sum_{i=1}^n \gamma_i\|Y_{i,\cdot}\|^2 \qquad \text{and} \qquad \|X\|_F^2 = \|Y\|_F^2 = \sum_{i=1}^n\|Y_{i,\cdot}\|^2 = 1,$$

respectively. As M is a "true 2D" mapping onto $\mathbb{R}^2$ it is of rank 2. Hence, to minimize the weighted average of the Laplacian eigenvalues, we put all weight onto the two smallest positive eigenvalues. Up to relabeling, we WLOG assume that γ_2, γ_3 are the two smallest nonzero eigenvalues of L. Then the objective is minimized at value $(\gamma_2+\gamma_3)/2$, realized by the transformation $v \mapsto (u_2(v), u_3(v))/\sqrt{2}$.

Verdict (Q3): basically identical, 3pts / 3pts.

Solution to problem 4. Let f_k be defined as in the problem. Let $S_r = \{u : d(u) = r\}$, and for $u \in S_r$, define the following quantities:

$$b(u) = |N(u)\cap S_{r-1}|, \qquad s(u) = |N(u)\cap S_r|, \qquad c(u) = |N(u)\cap S_{r+1}|.$$

Then $d(u), s(u), c(u)$ exhaust all neighbors of u, i.e., $d(u)+s(u)+c(u) = \deg(u) = d$. And for $r \geqslant 1$, $b(u) \geqslant 1$. For $1 \leqslant r \leqslant k-1$, we therefore have

$$(Af_k)(u) = b(u)(d-1)^{-(r-1)/2} + s(u)(d-1)^{-r/2} + c(u)(d-1)^{-(r+1)/2},$$

and dividing by $f_k(u) = (d-1)^{-r/2}$ gives

$$\frac{(Af_k)(u)}{f_k(u)} = b(u)\sqrt{d-1} + s(u) + \frac{c(u)}{\sqrt{d-1}} \geqslant \sqrt{d-1} + 0 + \frac{d-1}{\sqrt{d-1}} = 2\sqrt{d-1},$$

where the inequality minimizes the sum using $b(u) = 1, s(u) = 0$, and $c(u) = d - 1$.

For $r = k$, the neighbors in S_{k+1} contributes zero, so

$$\frac{(Af_k)(u)}{f_k(u)} = b(u)\sqrt{d-1} + s(u) \geqslant \sqrt{d-1}.$$

Finally, multiplying back by $f_k(u)$ and summing over all values/vertices yields

$$\langle f_k, Af_k\rangle \geqslant 2(d-1)^{1/2}\sum_{r=1}^{k-1}\sum_{u\in S_r} f_k(u)^2 + (d-1)^{1/2}\sum_{u\in S_k} f_k(u)^2 = 2(d-1)^{1/2}\|f_k\|_2^2 - (d-1)^{1/2}a_k$$

where $a_k = \sum_{u\in S_k} f_k(u)^2 = |S_k|(d-1)^{-k}$.

Now we take a look at the a_k's. Each vertex in S_{r+1} has at least one neighbor in S_r, and each vertex in S_r has at most $d-1$ neighbors in S_{r+1}. Hence $|S_{r+1}| \leqslant (d-1)|S_r|$, leading to

$$a_{r+1} = \frac{|S_{r+1}|}{(d-1)^{r+1}} \leqslant \frac{|S_r|}{(d-1)^r} = a_r,$$

so $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$. In particular, $a_k \leqslant \sum_{r=1}^k a_r/k = \|f_k\|_2^2/k$, so we have

$$\frac{\langle f_k, Af_k\rangle}{\|f_k\|_2^2} \geqslant 2(d-1)^{1/2} - (d-1)^{1/2}/k. \tag{*}$$

It remains to define a nice vector g. Let $R = \max_{u\in V\setminus\{v_0\}} d(u)$ and pick a vertex w with $d(w) = R$. Let $k = \lceil R/2\rceil - 1$ so that the two balls $B_k(v_0)$ and $B_k(w)$ are both vertex- and edge-disjoint. Define $f_k^{(v_0)}$ and $f_k^{(w)}$ analogously, with respect to this k. As k is fixed for the rest of the proof, we re-define them as f_{v_0} and f_w. Let $\mu(v_0) = \langle f_{v_0}, \mathbf{1}\rangle$ and $\mu(w) = \langle f_w, \mathbf{1}\rangle$, and set

$$g := \mu(w) f_{v_0} - \mu(v_0) f_w.$$

Then $\langle g, \mathbf{1}\rangle = 0$, and because the support of g is edge-disjoint,

$$\langle g, Ag\rangle = \mu(w)^2\langle f_{v_0}, Af_{v_0}\rangle + \mu(v_0)^2\langle f_w, Af_w\rangle$$

and

$$\|g\|_2^2 = \mu(w)\|f_{v_0}\|_2^2 + \mu(v_0)\|f_w\|_2^2.$$

Hence $\langle g, Ag\rangle/\|g\|_2^2$ as a weighted average of $\langle f_{v_0}, Af_{v_0}\rangle/\|f_{v_0}\|_2^2$ and $\langle f_w, Af_w\rangle/\|f_w\|_2^2$ is at least the minimum of the two. Finally, using (*) and the fact that $\langle g, \mathbf{1}\rangle = 0$, we conclude

$$\lambda_2(A) \geqslant \frac{\langle g, Ag\rangle}{\|g\|_2^2} \geqslant 2\sqrt{d-1} - \frac{\sqrt{d-1}}{k} = 2\sqrt{d-1} - \mathcal{O}\left(\frac{\sqrt{d-1}}{\max_{u\in V\setminus\{v_0\}} d(u)}\right). \qquad \square$$

Acknowledgments. I have consulted with a certain LLM for (the second) part of this problem.

Verdict (Q4). Same truncated-tree/Rayleigh idea as the sample solution, but different approaches, plus some typos. I guess 5.5 pts / 5.5 pts?

Solution to problem 5. Fix d, ϵ. As in Q4, fix a vertex v and let $d(u)$ be the shortest-path distance from v to u. For $k \in \mathbb{N}$, define $f_k^{(v)}(u) = f_k(u)$ as in Q_4: if $d(u) \leqslant k$, let $f_k(u) = (d-1)^{-d(u)/2}$, and zero otherwise. Hence (*) from Q4 holds verbatim.

Let k be such that $\sqrt{d-1}/k < \epsilon$. The main difference from Q4 is we consider a collection of balls instead of two. To this end, let $S \subset V$ be a set of vertices whose pairwise distances are at least $2(k+1)$. Then the balls $\{B_k(s)\}_{s\in S}$

centered at each vertex in S with radius k are vertex- and edge-disjoint, so the support of $\{f_k^{(s)}\}_{s\in S}$ is disjoint. Using greedy pairing, if $\alpha(d,2k+1)$ is an upper bound on the set size of a $(2k+1)$-ball in a d-regular graph, then $|S| \geqslant n/\alpha(d,2k+1)$. Expanding the search tree via BFS, we obtain crude bounds

$$\alpha(d,2k+1) \leqslant \begin{cases} 1 + d\sum_{j=0}^{2k}(d-1)^j \leqslant 4(d-1)^{2k+1} & d \geqslant 3 \\ 4k+3 & d = 2. \end{cases}$$

The rest is quite analogous. With abuse of notation, since k is fixed, we replace the subscript by the root node w.r.t. the function, i.e., replace $f_k^{(r)}$ by f_r. Then the disjoint conditions imply

$$\langle f_r, f_s\rangle = 0 \qquad \text{and} \qquad \langle f_r, Af_s\rangle = 0$$

for $r \neq s$. Let $\mathcal{W}$ be the subspace spanned by $\{f_s : s \in S\}$. For any $x \in \mathcal{W}$, write it in the decomposition $x = \sum_{s\in S} c_s f_s$. Then we have, by disjoint support and the weighted average argument as in Q4,

$$\frac{\langle x, Ax\rangle}{\|x\|_2^2} = \frac{\sum_{s\in S}|c_s|^2\langle f_s, Af_s\rangle}{\sum_{s\in S}|c_s|^2\|f_s\|_2^2} \geqslant \min_{s\in S}\frac{\langle f_s, Af_s\rangle}{\|f_s\|_2^2} \geqslant 2\sqrt{d-1}-\epsilon.$$

Now recall the "alternate" perspective of eigenvalues: $\lambda_m = \max_{\dim U=m}\min_{x\in U,\|x\|=1}\langle x, Ax\rangle$. Since $\mathcal{W}$ has dimension m, the inequality above implies $\lambda_m \geqslant 2\sqrt{d-1}-\epsilon$, so we are done! There are at least $|S| \geqslant n/\alpha(d,2k+1)$ eigenvalues of A that are at least $2\sqrt{d-1}-\epsilon$.

Verdict (Q5). I used the same idea as in Q4 by packing many disjoint k-balls to build a subspace and apply min-max. Similar to the sample solution conceptually although $\alpha(d,2k+1)$ is less tight in comparison. Still proved the intended claim. 2.5 pts / 2.5 pts.

Overall I think every problem has been solved correctly modulo small notational mistakes. Hence I'd venture to give myself full credit on HW1.