CS590.05 Homework 2

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Overall verdict: either 12/12 or 10.5/12 depending on how big the issue in Q5 reverse direction is. Personally I think it's relatively minor.

Solution to problem 1. By Cheeger's inequality, if $\Phi(G) \geqslant 1/4$ then $\Phi(G) \leqslant \sqrt{2(1-\lambda_2)}$ implies $\lambda_2 \leqslant 31/32$. Hence it suffices to prove that a uniformly random d-regular graph on n vertices has $\Phi(G) \geqslant 1/4$ w.h.p.

Fix a set $S \subset V$ with $|S| = s \leqslant n/2$. Let $E(S)$ be the number of edges with both endpoints in S and $E(S, S^c)$ the number of edges crossing S. Since $\mathrm{vol}(S) = 2E(S) + E(S, S^c)$, the bad event $E(S, S^c) \leqslant d/4 \cdot s$ implies $E(S) \geqslant 3/8 \cdot ds$. We let $t \geqslant 3/8 \cdot ds$, for example taking the ceiling.

Recall that we generate a random d-regular graph by giving each vertex d half-edges, and then take a uniformly random perfect matching on the dn half-edges. Among the ds half-edges in S, the number of ways to choose t disjoint unordered pairs is $N_S(t) = (ds)!/(2^t t!(ds-2t)!)$. For any fixed family of t disjoint pairs, the probability that all those pairs will appear in the random perfect matching equals

$$\frac{(dn-2t-1)!!}{(dn-1)!!} = \prod_{i=0}^{t-1} (dn - (2i+1))^{-1}.$$

Therefore, by union bound of such families,

$$\mathbb{P}(E(S) \geqslant t) \leqslant N_S(t) \cdot \frac{(dn-2t-1)!!}{(dn-1)!!} = \prod_{i=0}^{t-1} \binom{ds-2i}{2}\binom{dn-2i}{2}^{-1} \leqslant (s/n)^{2t}$$

where the last step used $\binom{a}{2}\binom{b}{2}^{-1} \leqslant (a/b)^2$ and that $dn - 2i \geqslant dn - 2t \geqslant dn/2$. With $t \geqslant 3/8 \cdot ds$ this cleanly gives

$$\mathbb{P}(E(S, S^c) \leqslant ds/4) \leqslant \mathbb{P}(E(S) \geqslant t) \leqslant \left(\frac{s}{n}\right)^{3/4 \cdot ds}.$$

Now let X be the number of bad sets S (i.e. with $E(S, S^c) \leqslant ds/4$). Then $\mathbb{E}X \leqslant \sum_{s=1}^{\lfloor n/2 \rfloor} \binom{n}{s} (s/n)^{3/4 \cdot ds}$. We split the sum into two ranges:

- If $1 \leqslant s \leqslant n/\log n$, using $\binom{n}{s} \leqslant (en/s)^s$, each summand is then at most

$$\binom{n}{s}\left(\frac{s}{n}\right)^{3/4 \cdot ds} \leqslant \left(e \cdot \left(\frac{s}{n}\right)^{3d/4-1}\right)^s \leqslant [\exp((\log n)^{-(3d/4-1)})]^s$$

  which $\to 0$ as $n \to \infty$, as $d \geqslant 3$ implies $3d/4 - 1 \geqslant 1/4$. Hence these terms collectively contribute at most $o(1)$.

- Now suppose $n/\log n \leqslant s \leqslant n/2$. Define the entropy function $H(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$. Writing $\alpha = s/n \in [1/\log n, 1/2]$ and using $\binom{n}{\alpha n} \leqslant \exp(nH(\alpha))$, each individual s-term is bounded by

$$\binom{n}{s}\left(\frac{s}{n}\right)^{3/4 \cdot ds} \leqslant \exp\left(n[H(\alpha) + \frac{3}{4} d\alpha \log \alpha]\right).$$

Let $f(\alpha) = H(\alpha) + 3/4 \cdot \alpha \log \alpha$. Then f is convex and hence the expression above is maximized at an endpoint. On one hand,

$$f(1/2) = \log 2 + \frac{3}{8} \cdot d \cdot \log(1/2) = (1 - 3d/8) \log 2 < 0$$

while at $\alpha = 1/\log n$,

$$f(1/\log n) \leqslant \frac{1}{\log n}(1 - (3d/4 - 1) \log\log n) = -\Omega(\log\log n/\log n).$$

Hence, every summand range in this case is at most $\exp(-\Omega(n))$ or $\exp(-\Omega(n \log\log n/\log n)$. Summing over at most n values (since $s \leqslant n/2$) gives a total of $o(1)$ contribution.

Combining both cases we see $\mathbb{E}X = o(1)$ which is exactly what we need to finish the proof.

Verdict: 3/3. Basically the intended proof. Small typos like $\prod_i \binom{ds-2i}{2}\binom{dn-2i}{2}^{-1}$ that misses a factor of 2.

Solution to problem 2. We inherit definitions and notations from the lecture. For $f : V(G_n) \to \mathbb{R}$ with $\langle f, \mathbf{1} \rangle = 0$, define

$$\tilde{f} : [0, n)^2 \to \mathbb{R} \qquad \text{by} \qquad \tilde{f}(x, y) = f(\lfloor x \rfloor, \lfloor y \rfloor).$$

Partition $[0, n)^2$ into unit cells $Q_{i,j} = [i, i+1) \times [j, j+1)$, so that

$$\langle \tilde{f}, \tilde{f} \rangle = \sum_{i,j} f(i,j)^2 = \langle f, f \rangle \qquad \text{and} \qquad \langle \tilde{f}, \mathbf{1} \rangle = \sum_{i,j} f(i,j) = \langle f, \mathbf{1} \rangle = 0.$$

Fix a cell $Q_{i,j}$ and define (indices modulo n hereafter)

$$A_{i,j} = (f(i,j) - f(i, j+i))^2, \qquad C_{i,j} = (f(i, j+i) - f(i, j+i+1))^2.$$

Inside $Q_{i,j}$, the line $x + y = i + j + 1$ splits it through a diagonal. On the lower triangle, $\tilde{f}(x, x+y) = f(i, j+i)$. On the upper, $\tilde{f}(x, x+y) = f(i, j+i+1)$. Hence

$$\int_{Q_{i,j}} (\tilde{f}(x,y) - \tilde{f}(x, x+y))^2 = \frac{1}{2} A_{i,j} + \frac{1}{2}(f(i,j) - f(i, j+i+1))^2 \leqslant \frac{3}{2} A_{i,j} + C_{i,j}$$

because $(u - v)^2 \leqslant 2(u - w)^2 + 2(w - v)^2$ with u, w, v defined as $f(i,j), f(i, j+i), f(i, j+i+1)$, respectively. Likewise, for

$$B_{i,j} = (f(i,j) - f(i+j, j))^2, \qquad D_{i,j} = (f(i+j, j) - f(i+j+1, j))^2$$

we have

$$\int_{Q_{i,j}} (\tilde{f}(x,y) - \tilde{f}(x+y, y))^2 \leqslant \frac{3}{2} B_{i,j} + D_{i,j}.$$

Combining these two inequalities and using the formula for $\langle f, \overline{L}_R f \rangle$ we obtain

$$\langle \tilde{f}, \overline{L}_R \tilde{f} \rangle \leqslant \frac{3}{8} \sum_{i,j} (A_{i,j} + B_{i,j}) + \frac{1}{4} \sum_{i,j} (C_{i,j} + D_{i,j}).$$

Now, group the undirected edges in G_n into four families:

- $(x, y) \leftrightarrow (x, y \pm x)$
- $(x, y) \leftrightarrow (x \pm y, y)$
- $(x, y) \leftrightarrow (x, y \pm 1)$

- $(x,y) \leftrightarrow (x \pm 1, y)$

Summing each family once over (i,j) yields precisely $\sum A_{i,j}, \cdots, \sum D_{i,j}$, respectively. Therefore,

$$\langle f, \overline{L}_G f \rangle = \frac{1}{8}\left(\sum_{i,j} A_{i,j} + \sum_{i,j} B_{i,j} + \sum_{i,j} C_{i,j} + \sum_{i,j} D_{i,j}\right).$$

Finally, since $1/4 \leqslant 3/8$, $\langle \tilde{f}, \overline{L}_R \tilde{f} \rangle \leqslant 3 \langle f, \overline{L}_G f \rangle$. Relate back to Rayleigh quotients and the proof is complete.

Verdict: 3/3. I expanded f to $\tilde{f}$ on $[0,n)^2$, compare Rayleigh numberators cell by cell, group edge families, and derive the intended bounds. Basically the intended solution.

Solution to problem 3. This one is simple :D Write $M = xI - A$. Brute force expansion gives

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{v \in V} M_{v,\sigma(v)}.$$

The diagonal entries of M are x, and an off-diagonal entry $M_{u,v}$ is -1 iff (u,v) is an edge of T and 0 otherwise. A permutation σ contributes a nonzero term only if, for every cycle $(v_1, v_2, \cdots, v_r)$ of σ with $r \geqslant 2$, the pairs (v_i, v_{i+1}) (with $v_{r+1} = v_1$) are edges of T. If $r \geqslant 3$, those pairs would form a simple cycle in T, which is impossible. Therefore, every nonzero term comes from a permutation whose cycle decomposition consists solely of fixed points and transpositions (2-cycles) with each transposition corresponding to an edge (u,v), and nothing else.

Because cycles in a permutation are vertex-disjoint, the set of transpositions in σ naturally forms a matching M. Conversely, any matching $M = \{(u_i, v_i)\}$ determines a unique permutation that swaps each u_i with the corresponding v_i, fixing others. For this σ_M with $|M| = k$, the contribution to the determinant is $(-1)^k x^{n-2k}$. Summing over all matchings (over all k and all matchings of size k) gives

$$\det(xI - A) = \sum_{k \geqslant 0} (\text{\# of matchings of size } k)(-1)^k x^{n-2k} = \mu_T(x).$$

Verdict: 3/3. Same overall argument: only fixwed points and transpositions can contribute to a tree, and transpositions correspond bijectively to matchings.

Solution to problem 4. Skipped due to insufficient time. Will read sample solution. □

Verdict: 0/0.

Solution to problem 5. For the forward direction, let q be a degree $n-1$ intelacer for both f and g. Write the zeroes of q as $\gamma_1 > \cdots > \gamma_{n-1}$. By assumption, f has exactly one zero in each of the n intervals $(-\infty, \gamma_{n-1}), \cdots, (\gamma_2, \gamma_1), (\gamma_1, \infty)$ and likewise for g. At each γ_k, the signs of $f(\gamma_k)$ alternates. Likewise for g. Furthermore, $\operatorname{sgn}(f(\gamma_k)) = \operatorname{sgn}(g(\gamma_k)) = (-1)^k$ for each k, as they both have positive leading coefficients. Hence, for every $t \in [0,1]$,

$$\operatorname{sgn} h_t(\gamma(k)) = \operatorname{sgn}(tf(\gamma_k) + (1-t)g(\gamma_k)) = (-1)^k.$$

Consecutive values have opposite signs, so h_t has one real root in each of the n intervals above, and because $\deg h_t = n$, the forward direction is proven.

Conversely, write the roots of g as $\beta 1 > \beta 2 > \cdots > \beta_n$. For $c \geqslant 0$, set $p_c(x) = f(x) + cg(x) = (1+c)h_{1/(1+c)}(x)$. Then by assumption each p_c is real rooted.

We first claim that for suffciently large c, the polynomial p_c has exactly one root in each interval (β_{j+1}, β_j), and none outside (β_n, β_1). To see this, observe that as $c \to \infty$, $c^{-1}p_c = g + c^{-1}f \to g$ uniformly on compact sets avoiding $\{\beta_j\}$. Hence, for a large c, each root of p_c lies sufficiently close to some β_j, and all roots are bounded (because f and g have the leading coefficient and hence the same sign as $x \to \pm\infty$). Thus, for large c, each interval (β_{j+1}, β_j) contains exactly one root.

We also claim that for any $c \geqslant 0$, each intermval (β_{j+1}, β_j) must contain at least one root of p_c. To see this, fix j and let $N_j(c)$ be the number of roots p_c has that is in (β_{j+1}, β_j) (so we show $N_j(c) = 1$). As c varies, roots of p_c move continuously and may leave/enter (β_{j+1}, β_j) only by crossing one of the endpoints. But $p_c(\beta_k) = f(\beta_k)$ is independent of c as $g(\beta_k) = 0$. Hence no such crossing is possible.

In particular, taking $c = 0$ shows that between every pair (β_{j+1}, β_j), there is a root of f. Since f has degree n, this forces exactly one root between each consecutive pair, i.e., f and g interlace, as claimed.

Verdict: 3/3. My solution is definitely more verbose than needed, but both directions are essentially correct. I do notice that I argued in the reverse direction that for large c, $p_c = f + cg$ has exactly one root in each open interval between consecutive roots of g and none outside (β_n, β_1). This is off by 1 given that there are n roots total. I think this doesn't constitute a major issue but one could argue that this solution is 1.5/3.