Homework 2 Solution

Release date: Sep. 26, Due date: Oct. 3.

Guidelines: Submit your self-grade in pdf format on Gradescope by **5pm on Oct. 3**. Please compare your solutions with the posted ones and assign yourself points honestly:

- Full credit: essentially correct
- 50% credit: good progress but errors/incomplete
- 25% credit: some relevant ideas only
- No credit: no real attempt

If you take partial credit, briefly note why.

Q 1 (Random graph expansion, 3 pts)**.** Prove the following theorem about expansion of d-regular graphs:
For every fixed $d \geq 3$, a uniformly random d-regular graph on n vertices is a one-sided $\frac{31}{32}$-expander with probability $1 - o(1)$.

Hint: Prove that w.h.p. the graph conductance is at least $\frac{1}{4}$.

Solution

We consider the configuration model where a d-regular graphs are sampled by sampling a random matching over dn half edges. If a d-regular G has conductance $< \frac{1}{4}$, then there exists some $S \subseteq V$ with $|S| = k \leq n/2$ and some $T \subseteq V$ with $|T| < kd/4$, such that $N(S) \subseteq S \cup T$, i.e. all kd half-edges of S connect into $S \cup T$.

In a random d-regular graph, for a specific pair S and T, the probability that this bad event happens is

$$\Pr[N(S) \subseteq S \cup T] \;\leq\; \left(\left(1 + \frac{d}{4}\right)\frac{k}{n}\right)^{kd} = \left(\frac{O_d(k)}{n}\right)^{kd}.$$

There are at most

$$\binom{n}{k}\binom{n}{kd/4}$$

choices of S, T. So by a union bound, the total failure probability is

$$\leq \sum_{k=1}^{n/2} \binom{n}{k}\binom{n}{kd/4}\left(\frac{O_d(k)}{n}\right)^{kd}.$$

Using $\binom{n}{m} \leq (en/m)^m$, the kth summand is

$$\leq \left(\frac{en}{k}\right)^{k}\left(\frac{4en}{kd}\right)^{kd/4}\left(\frac{O_d(k)}{n}\right)^{kd}.$$

By our choice of d, each summand is bounded by $\left(\frac{O_d(k)}{n}\right)^{k}$. So the whole sum is $o(1)$.

Therefore w.h.p. no set S has $|N(S)| < kd/4$. Hence for all S with $|S| \leq n/2$,

$$|N(S)| \;\geq\; \tfrac{d}{4}|S|,$$

so the graph has conductance $\geq 1/4$, which implies it is also a one-sided $31/32$-expander. □

Q 2 (MGG expanders, 3 pts)**.** Recall that in the analysis of Margulis-Gabber-Galil expanders G_n we defined the graph R_n. Prove the following inequality on the spectral gaps of G_n and R_n:

$$\gamma(G_n) \geq \frac{1}{3}\gamma(R_n).$$

Hint: For every function $f : (\mathbb{Z}/n\mathbb{Z})^2 \to \mathbb{R}$ in G_n, consider $\tilde{f} : [0, n)^2 \to \mathbb{R}$ such that $\tilde{f}(x, y) = f(\lfloor x \rfloor, \lfloor y \rfloor)$.

Solution from notes by Luca Trevisan

Let f be a function such that

$$\gamma(G_n) = \frac{\sum_{c\in\mathbb{Z}_n^2}\left(|f(c) - f(S(c))|^2 + |f(c) - f(T(c))|^2 + |f(c) - f(c + (0, 1))|^2 + |f(c) - f(c + (1, 0))|^2\right)}{8\sum_{c\in\mathbb{Z}_n^2} f^2(c)}.$$

For $(x, y) \in [0, n)^2$, define $\lfloor x, y\rfloor := (\lfloor x\rfloor, \lfloor y\rfloor)$. Extend f to $\tilde{f} : [0, n)^2 \to \mathbb{R}$ by setting $\tilde{f}(z) = f(\lfloor z\rfloor)$. Thus $\tilde{f}$ is constant on each unit square with value given at its lower-left corner.

It is immediate that

$$\int_{[0,n)^2} \tilde{f}^2(z)\, dz = \sum_{c\in\mathbb{Z}_n^2} f^2(c),$$

so up to a factor of 2, the denominators of the Rayleigh quotients of f and $\tilde{f}$ agree.

We now compare numerators. For every $z \in [0, 1)^2$, the point $\lfloor S(z)\rfloor$ equals either $S(\lfloor z\rfloor)$ or $S(\lfloor z\rfloor) + (0, 1)$, and similarly for $T(z)$. Hence the numerator of $\tilde{f}$ is

$$\sum_{c=(a,b)\in\mathbb{Z}_n^2} \int_{[a,a+1)\times[b,b+1)} \left(|\tilde{f}(z) - \tilde{f}(S(z))|^2 + |\tilde{f}(z) - \tilde{f}(T(z))|^2\right) dz.$$

By symmetry, this equals

$$\frac{1}{2}\sum_{c\in\mathbb{Z}_n^2}\left(|f(c) - f(S(c))|^2 + |f(c) - f(S(c) + (0, 1))|^2 + |f(c) - f(T(c))|^2 + |f(c) - f(T(c) + (1, 0))|^2\right).$$

Applying the inequality

$$|\alpha - \gamma|^2 \;\leq\; 2|\alpha - \beta|^2 + 2|\beta - \gamma|^2,$$

we can bound the numerator above by

$$\frac{1}{2}\sum_{c\in\mathbb{Z}_n^2}\left(3|f(c) - f(S(c))|^2 + 3|f(c) - f(T(c))|^2 + 2|f(c) - f(c + (0, 1))|^2 + 2|f(c) - f(c + (1, 0))|^2\right).$$

This is at most $\frac{3}{2}$ times the numerator of the Rayleigh quotient of f. □

Q 3. (Matching polynomials of trees, 3 pts) Given a tree T, prove that its matching polynomial $\mu_T(x)$ is equal to its characteristic polynomial $\det(xI - A)$ where A is its adjacent matrix. (From this one can deduce that if T has maximum degree d, then the roots of $\mu_T(x)$ are in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.)

Solution

By definition, for any graph $G = (V, E)$ and vertex $u \in V$, the matching polynomial satisfies the recurrence relation

$$\mu_G(x) \;=\; x\,\mu_{G\setminus u}(x) \;-\; \sum_{uv\in E} \mu_{G\setminus\{u,v\}}(x),$$

where $G \setminus u$ is obtained by removing vertex u in G, and $G \setminus \{u, v\}$ by removing $\{u, v\}$ from G.

This comes from considering whether a matching uses an edge incident to u or not.

On the other hand, doing the Laplace expansion of $\det(xI - A_T)$ along the row/column of u gives

$$\phi_T(x) \;=\; x\,\phi_{T\setminus u}(x) \;-\; \sum_{uv\in E(T)} \phi_{T\setminus\{u,v\}}(x),$$

where $\phi_T(x) = \det(xI - A_T)$.

By induction on $|V(T)|$: the base case $|T| = 1$ is trivial ($\mu_T(x) = x = \phi_T(x)$), and if the equality holds for smaller trees, then the identical recurrences imply $\mu_T(x) = \phi_T(x)$. $\square$

Q 4. (Optional question, 0 pt) Given a d-regular graph $G = (V, E)$ and a vertex $v_1 \in V$, we can construct its path tree T_{v_1} as follows:

The vertices in T_{v_1} are all paths in G that start at v_1 and do not visit any vertices twice. Two paths are connected iff one extend the others by one step.

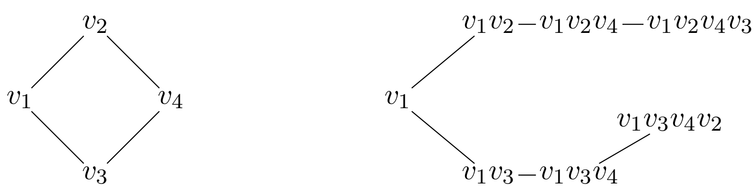

Figure 1: Graph G (left) and its path tree T_{v_1} (right).

Let $G \setminus v_1$ denote the graph obtained by removing v_1 from G, and $T_{v_1} \setminus v_1$ denote the graph obtained by removing v_1 from T_{v_1}.

Prove that

$$\frac{\mu_G(x)}{\mu_{G\setminus v_1}(x)} = \frac{\mu_{T_{v_1}}(x)}{\mu_{T_{v_1}\setminus v_1}(x)}.$$

(Using this statement, one can deduce that roots of $\mu_G(x)$ are a subset of roots of $\mu_{T_{v_1}}(x)$. As a result roots of $\mu_G(x)$ are in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.)

Hint: Induce on the size of the graph G. Note that $T_{v_1} \setminus v_1$ can be viewed as the union of a couple path trees of $G \setminus v_1$.

Solution from notes by Dan Spielman

If G is a tree, then $T_{v_1}(G) \cong G$ and the equality holds. As the only graphs with fewer than 3 vertices are trees, the claim holds for those as well. We proceed by induction on $|V|$.

By the recurrence relation of matching polynomials

$$\frac{\mu_G(x)}{\mu_{G\setminus v_1}(x)} = \frac{x\,\mu_{G\setminus v_1}(x) - \sum_{v_1v_2\in E}\mu_{G\setminus v_1\setminus v_2}(x)}{\mu_{G\setminus v_1}(x)} = x - \sum_{v_2\sim v_1} \frac{\mu_{G\setminus v_1\setminus v_2}(x)}{\mu_{G\setminus v_1}(x)}.$$

Applying the inductive hypothesis to $G \setminus v_1$ gives

$$\frac{\mu_G(x)}{\mu_{G\setminus v_1}(x)} = x - \sum_{v_2\sim v_1} \frac{\mu_{T_{v_2}(G\setminus v_1)\setminus v_2}(x)}{\mu_{T_{v_2}(G\setminus v_1)}(x)}. \tag{1}$$

Now observe

$$T_{v_1}(G) \setminus v_1 \;=\; \bigsqcup_{v_1v_2\in E} T_{v_2}(G \setminus v_1), \quad \mu_{T_{v_1}(G)\setminus v_1}(x) \;=\; \prod_{v_1v_2\in E} \mu_{T_{v_2}(G\setminus v_1)}(x).$$

For a neighbor $v_1 \sim v_2$, the subtree $T_{v_1}(G) \setminus v_1 \setminus v_1 v_2$ decomposes as

$$\Big(\bigsqcup_{v_1 c \in E,\, c \neq v_2} T_c(G \setminus v_1) \Big) \sqcup (T_{v_2}(G \setminus v_1) \setminus v_2),$$

so

$$\frac{\mu_{T_{v_1}(G) \setminus v_1 \setminus v_1 v_2}(x)}{\mu_{T_{v_1}(G) \setminus v_1}(x)} = \frac{\mu_{T_{v_2}(G \setminus v_1) \setminus v_2}(x)}{\mu_{T_{v_2}(G \setminus v_1)}(x)}.$$

Plugging this into (1),

$$\frac{\mu_G(x)}{\mu_{G \setminus v_1}(x)} = x - \sum_{v_2 \sim v_1} \frac{\mu_{T_{v_1}(G) \setminus v_1 \setminus v_1 v_2}(x)}{\mu_{T_{v_1}(G) \setminus v_1}(x)} = \frac{x\, \mu_{T_{v_1}(G) \setminus v_1}(x) - \sum_{v_2 \sim v_1} \mu_{T_{v_1}(G) \setminus v_1 \setminus v_1 v_2}(x)}{\mu_{T_{v_1}(G) \setminus v_1}(x)}.$$

By the recurrence relation, the numerator is exactly $\mu_{T_{v_1}(G)}(x)$. Thus

$$\frac{\mu_G(x)}{\mu_{G \setminus v_1}(x)} = \frac{\mu_{T_{v_1}(G)}(x)}{\mu_{T_{v_1}(G) \setminus v_1}(x)}.$$

□

Q 5 (Interlacing polynomials, 3 pts)**.** Prove the following lemma we used to analyse the Marcus–Spielman–Srivastava bipartite Ramanujan graphs: Let f, g be univariate polynomials of the same degree with positive leading coefficients. Then f, g have a common interlacing if and only if for all convex combinations $h_t(x) = tf(x) + (1-t)g(x)$ where $t \in [0,1]$ is real-rooted.

Solution from notes by Jan Vondrák

($\Rightarrow$) Suppose f, g have a common interlacing with separating points $\mu_n \leq \cdots \leq \mu_0$. On each interval (μ_i, μ_{i-1}), f and g alternate signs, hence so does $h_t(x)$. Thus h_t has a root in every such interval, so h_t is real-rooted. For multiple roots, perturb f, g slightly and pass to the limit.

($\Leftarrow$) Suppose h_t is real-rooted for all $t \in [0,1]$. Let $\lambda_k(t)$ be the k-th largest root of h_t. As t varies, $\lambda_k(t)$ moves continuously. If f and g had no common interlacing, then two roots would have to cross, which would force a shared root of f and g. If shared roots exist, factor them out and apply the argument to the remaining factors. Hence f and g admit a common interlacing.

□