Expanders HW3

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3/3. Idea is mostly correct. Set up a bad set and drive probability of bad set to a small quantity.

Solution to problem 1. Fix $S \subset L$ with $\epsilon = |S|/|L|$ and $C > 0$. Let T be the bad set

$$T = \{v \in R : |N(v) \cap S|/|N(v)| > (1+C)\epsilon\}.$$

Since every $v \in R$ has the same degree d_R,

$$|E(S,T)| = \sum_{v \in T} |N(v) \cap S| > (1+C)\epsilon d_R |T| = (1+C)\epsilon \mathrm{vol}(T).$$

From the mixing lemma from lec2[?],

$$|E(S,T)| \leqslant \frac{\mathrm{vol}(S)\mathrm{vol}(T)}{\mathrm{vol}(T)} + \lambda\sqrt{\mathrm{vol}(S)\mathrm{vol}(T)}.$$

Because the graph is bipartite and biregular, the left/right degrees satisfy $d_L|L| = d_R|R|$. Thus $\mathrm{vol}(S)/\mathrm{vol}(V) = d_L|S|/(d_R|R|) = |S|/|L| = \epsilon$. Dividing by $\mathrm{vol}(T)$, then using the two equations/inequalities above gives

$$(1+C)\epsilon < \epsilon + \lambda\sqrt{\mathrm{vol}(S)/\mathrm{vol}(T)} \qquad \Longrightarrow \qquad C\epsilon\sqrt{\mathrm{vol}(T)} < \lambda\sqrt{\mathrm{vol}(S)}.$$

Finally, using $\mathrm{vol}(S) = \epsilon \mathrm{vol}(V)$ we get

$$C^2\epsilon^2\mathrm{vol}(T) < \lambda^2\epsilon\mathrm{vol}(V) \qquad \Longrightarrow \qquad \frac{\mathrm{vol}(T)}{\mathrm{vol}(V)} < \frac{\lambda^2}{\epsilon C^2}.$$

This is precisely what we want, since $\mathrm{vol}(T)/\mathrm{vol}(V)$ is precisely $|T|/|R|$.

3/3. Observe $\mu_G(i,j) = \pi_0(i)P(i,j)$, from there compute $\mathbb{E}[f(X)g(Y)] = \langle f, Pg\rangle$, and conclude the claim.

Solution to problem 2. From definition, $\mu_G(i,j) = \pi_0(i)P(i,j)$ so that X, Y have marginals π_0, and for any $f, g : V \to \mathbb{R}$, we have

$$\mathbb{E}_{\mu_G}[f(X)g(Y)] = \sum_{i,j} f(i)g(j)\pi_0(i)P(i,j) = \langle f, Pg\rangle_{\pi_0}.$$

Consequently $\mathbb{E}_{\mu_G} f(X) = \langle f, \mathbf{1}\rangle_{\pi_0}$. We normalize the variational problem by defining $\tilde{f} = f - \langle f, \mathbf{1}\rangle_{\pi_0}$ so that $\langle \tilde{f}, \mathbf{1}\rangle_{\pi_0} = 0$, and likewise define $\tilde{g}$. Then

$$\mathbb{E}_{\mu_G}[f(x)g(Y)] - \mathbb{E}_{\mu_G}[f(X)]\mathbb{E}_{\mu_G}[g(Y)] = \langle \tilde{f}, P\tilde{g}\rangle_{\pi_0}.$$

Therefore

$$|\rho| = \sup_{\substack{\tilde{f}\perp\mathbf{1}, \tilde{g}\perp\mathbf{1} \\ \tilde{f},\tilde{g}\neq 0}} \frac{|\langle \tilde{f}, P\tilde{g}\rangle_{\pi_0}|}{\|\tilde{f}\|_{\pi_0}\|\tilde{g}\|_{\pi_0}}$$

which is equal to the operator norm of P on the subspace orthogonal to $\mathbf{1}$. Finally, because the chain is reversible, P is self adjoint, so

$$\langle f, Pg\rangle_{\pi_0} = \sum_{i,j} f(i)g(j)\pi_0(i)P(i,j) = \sum_{i,j} f(j)g(i)\pi_0(j)P(i,j) = \langle Pf, g\rangle_{\pi_0}.$$

Then the operator norm of P on $\mathbf{1}^{\perp}$ equals its largest singular value there, which equals the maximum absolute eigenvalue on $\mathbf{1}^{\perp}$. Hence $|\rho| = |\lambda|_2(P)$, as claimed.

6/6. Essentially correct up to notational slips.

Solution to problem 3. (1) Fixed ordered indices (i,j). If $\{i,j\} \notin X(1)$, then no d-face contains both i,j, so $\mu_X((Y_0,Y_1)=(i,j)) = 0 = \mu_G(i,j)$. We therefore assume $\{i,j\} \in X(1)$. Then

$$\mathbb{P}_{\mu_X}[(Y_0,Y_1)=(i,j)] = \sum_{\substack{a\in X(d)\\ \{i,j\}\subset a}} \frac{\pi_d(a)}{(d+1)!} \times \#\{\text{orderings of } a \text{ with } Y_0 = i, Y_1 = j\}.$$

For any such a, the remaining $d-1$ vertices can be ordered arbitrarily, giving $(d+1)!$ orderings. Hence,

$$\mathbb{P}_{\mu_X}[(Y_0,Y_1)=(i,j)] = \frac{(d-1)!}{(d+1)!}\sum_{\{i,j\}\subset a}\pi_d(a) = \frac{1}{d(d+1)}\sum_{\{i,j\}\subset a}\pi_d(a).$$

On the other hand, by definition, $\pi_1(\{i,j\}) = \frac{2}{d(d+1)}\sum_{\{i,j\}\subset a}\pi_d(a)$, so

$$\mu_G(i,j) = \frac{\pi_1(\{i,j\})}{2} = \frac{1}{d(d+1)}\sum_{\{i,j\}\subset a}\pi_d(a)$$

Therefore, for all ordered pairs (i,j), the marginal (Y_0,Y_1) under μ_X equals μ_G.

(2) Fix $v\in X(0)$ and an ordered d-tuple $(y_1,\cdots,y_d)$ with $\tau = \{y_i\}\in X_v(d-1)$, so $a = \tau\cup\{v\}\in X(d)$. Then,

$$\mathbb{P}_{\mu_X}[(Y_1,\cdots,Y_d)=(y_1,\cdots,y_d)\mid Y_0 = v] = \frac{\mu_X(v,y_1,\cdots,y_d)}{\mathbb{P}_{\mu_X}[Y_0=v]}$$

where the numerator equals $\pi_d(a)/(d+1)!$. For the denominator we sum over all $d!$ orderings of each a that contains v:

$$\mathbb{P}_{\mu_X}[Y_0=v] = \sum_{v\in a}\frac{d!\pi_d(a)}{(d+1)!} = \frac{1}{d+1}\sum_{v\in a}\pi_d(a).$$

Then the condition equals $\pi_d(a)/[d!\sum_{v\in a'}\pi_d(a')] = \pi_v(\tau)/d! = \mu_{X_v}(y_1,\cdots,y_d)$. This establishes the claim.

(3) From (2), conditioning on $Y_0 = v$ gives the distribution μ_{X_v} on the link X_v with $(d-1)$-face weights π_v induced by π_d. We apply part (1) to the $(d-1)$-dimensional complex X_v. Then, the marginal of the first two coordinates under μ_{X_v} equals the edge distribution μ_{G_v} of the 1-skeleton $G_v = X_v(1)$ with edge weights $\pi_1^{(v)}$ induced by π_v. Concretely, for $(u,w)\in X_v(1)$,

$$\nu[(Y_1,Y_2)=(u,w)] = \frac{(d-2)!}{d!}\sum_{\substack{\tau\in X_v(d-1)\\ \{u,w\}\subset\tau}}\pi_v(\tau) = \frac{1}{2}\pi_1^{(v)}(\{u,w\}) = \mu_{G_v}(u,w).$$

Thus $\nu = \mu_{G_v}$. By the previous problem, for any weighted graph H with law μ_H and random walk transition matrix P_H, one has $|\rho(\mu_H)| = |\lambda|_2(P_H)$. Using $H = G_v$ gives $|\rho(\nu)| = |\lambda|_2(P_v)$.