CS590 Homework 4

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Problem 1:

(1) Part 1: 4/4. I set up A_v, decompose $f\,|_v = (Af)(v) \cdot \mathbf{1} + g_v$ and apply the variational characterization in the links. Nothing special here.

(2) Part 2: 1/1. Just induction.

(3) Part 3: 0.5/1. Minor reasoning mistake handling $i = k - 1$ but otherwise correct argument.

Problem 2:

(1) Part 1: 1.5/3. Oversimplified the argument for the second term by using a triangle-ineq type of argument; what actually goes on is a majority-vs-sample comparison. The control over Q_3 is somewhat flawed too by the same token.

(2) Part 2: 3/3 skipped.

Overall: 10/12.

Solution to problem 1. **Part 1.** We follow the base-case proof of the trickle-down theorem as in lecture 10 and track the smallest eigenvalue. For each $v \in X(0)$, let A_v be the random walk on the 1-skeleton of the link X_v. Recall that (i) for every $f : X(0) \to \mathbb{R}$,

$$\langle f, Af \rangle = \mathbb{E}_{v \sim \pi_0}[\langle f\,|_v, A_v f\,|_v \rangle_v]$$

and (ii) if we decompose $f\,|_v = (Af)(v) \cdot \mathbf{1} + g_v$ with $g_v \perp \mathbf{1}$ in the link, then

$$\mathbb{E}_{v \sim \pi_0} \|f\,|_v\|_v^2 = \|f\|^2, \qquad \text{and} \qquad \langle f\,|_v, A_v f\,|_v \rangle_v = \langle g_v, A_v g_v \rangle_v + (Af)(v)^2.$$

Following the theorem assumption, let $\alpha = \lambda_{\min}(A)$ and pick an eigenfunction f with $Af = \alpha f$. Using above and the variational characterization of the smallest eigenvalue in each link,

$$\alpha \|f\|^2 = \langle f, Af \rangle = \mathbb{E}_v[\langle g_v, A_v g_v \rangle_v + (Af)(v)^2] \geqslant \lambda_{\min} \mathbb{E}_v \|g_v\|_v^2 + \|Af\|^2.$$

Since $\|f\,|_v\|_v^2 = \|g_v\|_v^2 + (Af)(v)^2$ and $\mathbb{E}_v \|f\,|_v\|_v^2 = \|f\|^2$, we have $\mathbb{E}_v \|g_v\|_v^2 = \|f\|^2 - \|Af\|^2$. Therefore

$$\alpha \|f\|^2 \geqslant \lambda_{\min}(\|f\|^2 - \alpha^2 \|f\|^2) + \alpha^2 \|f\|^2,$$

from which the result follows from dividing by $\|f\|^2 > 0$, $1 - \lambda_{\min} > 0$, and rearranging.

Part 2. We induct on $m = d - i - 1$ where $i < d - 2$. When $m = 1$, the 1-skeleton is a graph, so the random walk matrix has $\lambda_{\min} \geqslant -1$. From step m to $m + 1$, we let Y be any $(m + 1)$-dimensional complex. For every $v \in Y(0)$, the link Y_v is m-dimensional, so by IH, $\lambda_{\min}(Y_v^{\leqslant 1}) \geqslant -1/m$. Applying part 1 to the 2-skeleton $Y^{\leqslant 2}$ yields the inductive step:

$$\lambda_{\min}(Y^{\leqslant 1}) \geqslant \frac{-1/m}{1-(-1/m)} = -\frac{1}{m+1}.$$

Step 3. Let $1 \leqslant k \leqslant d$. For any $i \leqslant k-2$ and $\sigma \in X^{\leqslant k}(i)$, the 1-skeleton of $(X^{\leqslant k})_\sigma$ equals that of X_σ. From lecture 10, every link graph X_σ has $\lambda_2(X_\sigma) \leqslant \lambda$ since X is a d-dimensional one-sided λ-expander. On the lower side, by part 2 applied to X,

$$\lambda_{\min}(X_\sigma) \geqslant -\frac{1}{d-i-1} > -\frac{1}{d-k+1}.$$

Aggregating both bounds gives the desired claim.

Solution to problem 2. **Step 1.** Let $\epsilon = \mathbb{P}_{(s_1,s_2)\sim D}[f_{s_1}|_{s_1\cap s_2} \not\equiv f_{s_2}|_{s_1\cap s_2}]$ and let the lemma's constant be $C>0$, i.e.,

$$\mathbb{E}_t[\mathbb{P}_{s\supset t}[f_s|_{s\setminus t} \not\equiv g_{-t}|_{s\setminus t}]] \leqslant C\epsilon.$$

Define the three (LHS) quantities to be minimized as $Q_1(t_1), Q_2(t_1,t_2)$, and $Q_3(t_1,t_2)$, respectively. We show that $\mathbb{E}[Q_1] \leqslant C\epsilon, \mathbb{E}[Q_2] \leqslant 2C\epsilon$, and $\mathbb{E}[Q_3] \leqslant 2\epsilon$. Choosing (t_1,t_2) to minimize Q_1,Q_2,Q_3 then gives $Q_i \leqslant 5C\epsilon = \mathcal{O}(\epsilon)$.

The first term is already done. For Q_2, by triangle inequality inside the probability,

$$Q_2(t_1,t_2) \leqslant \mathbb{P}_{s\supset t_1}[g_{-s\setminus t_1}|_{t_1} \not\equiv f_s|_{t_1}] + \mathbb{P}_{s\supset t_1}[f_s|_{t_1} \not\equiv g_{-t_2}|_{t_1}].$$

The first term can be averaged over t_1; using bijection between $t_1 \subset s$ and $t = s\setminus t_1$ gives

$$\mathbb{E}_{t_1}\mathbb{P}_{s\supset t_1}[g_{-s\setminus t_1}|_{t_1} \not\equiv f_s|_{t_1}] = \mathbb{E}_t\mathbb{P}_{s\supset t}[g_{-t}|_{s\setminus t} \not\equiv f_s|_{s\setminus t}] \leqslant C\epsilon$$

whereas for the second term, averaging over disjoint t_1,t_2 and using $s = t_1 \cup t_2$ gives a similar bound. Hence $\mathbb{E}[Q_2] \leqslant 2C\epsilon$. The third term can be bounded likewise. Finally, because $\mathbb{E}[Q_1+Q_2+Q_3] \leqslant 5C\epsilon = \mathcal{O}(\epsilon)$, there exist t_1,t_2 satisfying this property.

Step 2. From the hint,

$$\mathbf{1}\{f_s \not\equiv g|_s\} \leqslant 2\mathbb{P}_{t\subset s,|t|=k/2}[f_s|_t \not\equiv g|_t].$$

Averaging over uniform s, then swapping the sampling order (first let $t \sim \binom{[n]}{k/2}$, then letting $s \sim$ uniform among the $\binom{[n]}{k}$ supersets of t) gives

$$\mathbb{P}_s[f_s \not\equiv g|_s] \leqslant 2\mathbb{E}_t\mathbb{P}_{s\supset t}[f_s|_t \not\equiv g|_t].$$

Now we fix t and write $X = s\setminus t$, so $|X| = k/2$. For this specific pair (s,t), basic set theoretic operations give

$$\mathbf{1}\{f_s|_t \neq g|_t\} \leqslant \mathbf{1}\{f_s|_t \neq g_{-X}|_t\} + \mathbf{1}\{g_{-X}|_{t\setminus t_1} \neq g_{-t_1}|_{t\setminus t_1}\} + \mathbf{1}\{g_{-X}|_{t\cap t_1} \neq g_{-t_2}|_{t\cap t_1}\}.$$

We now take expectations of the three terms citing part 1.

For the first term, by symmetry of joint sampling,

$$\mathbb{E}_t\mathbb{P}_{s\supset t}[f_s|_t \neq g_{-X}|_t] = \mathbb{E}_t\mathbb{P}_{s\supset t}[f_s|_{s\setminus t} \neq g_{-t}|_{s\setminus t}] \leqslant \mathcal{O}(\epsilon)$$

by the lemma. For the second term, since disagreement on subset $t\setminus t_1$ is dominted by disagreement on the whole $s\setminus t_1$, by part 1, we have

$$\mathbb{E}_t\mathbb{P}_{s\supset t}[g_{-X}|_{t\setminus t_1} \neq g_{-t_1}|_{t\setminus t_1}] \leqslant \mathbb{P}_{s\supset t_1}[g_{-(s\setminus t_1)}|_{s\setminus t_1} \neq g_{-t_1}|_{s\setminus t_1}] \leqslant \mathcal{O}(\epsilon).$$

Finally, the last term can be bounded analogously: disagreement on $t \cap t_1$ is dominated by that on all of t_1:

$$\mathbb{E}_t \mathbb{P}_{s \supset t}[g_{-X}\,|_{t \cap t_1} \neq g_{-t_2}\,|_{t \cap t_1}] \leqslant \mathbb{P}_{s \supset t_1}[g_{-(s \setminus t_1)}\,|_{t_1} \neq g_{-t_2}\,|_{t_1}] \leqslant \mathcal{O}(\epsilon),$$

by part 1 again. Combining the three inequalities completes the proof.