Homework 4 Solution

Release date: Nov. 1, Due date: Nov. 8.

Guidelines: Submit your self-grade in pdf format on Gradescope by **5pm on Nov. 8**. Please compare your solutions with the posted ones and assign yourself points honestly:

- Full credit: essentially correct
- 50% credit: good progress but errors/incomplete
- 25% credit: some relevant ideas only
- No credit: no real attempt

If you take partial credit, briefly note why.

Q 1 (From one-sided HDXs to two-sided HDXs, 6 pts)**.** In this problem we will prove that given a d-dimensional one-sided λ-expander X and an integer $1 \le k < d$, its k-skeleton $X^{\le k}$, which is obtained by removing from X all faces of dimension $> k$ and keeping the same distributions over the other faces, is a k-dimensional two-sided γ-expander where $\gamma = \max(\lambda, \frac{1}{d-k+1})$. We break the proof in two steps.

1. (4 pts) First show a variant of the trickle-down theorem:

   Theorem 0.1. Let X be a 2-dimensional simplicial complex with weight w_2. Suppose that for every $v \in X(0)$, the 1-skeleton of the link X_v has its smallest eigenvalue $\lambda_{\min}(X_v) \ge \lambda_{\min}$, then the 1-skeleton of X has its smallest eigenvalue $\lambda_{\min}(X^{\le 1}) \ge \frac{\lambda_{\min}}{1-\lambda_{\min}}$.

2. (1 pt) Use the theorem above to prove that in a d-dimensional X, for any $i < d-2$ and $\sigma \in X(i)$, the 1-skeleton of the link X_σ has its smallest eigenvalue $\ge -\frac{1}{d-i-1}$.

   (Note that the smallest eigenvalue of any random walk matrix is at least -1.)

3. (1 pt) Finish the proof that $X^{\le k}$ is a k-dimensional two-sided $\max(\lambda, \frac{1}{d-k+1})$-expander.

Solution

1. Let X be a 2-dimensional simplicial complex with vertex weights induced by w_2. Suppose that for every $v \in X(0)$, the 1-skeleton of the link X_v has smallest eigenvalue $\lambda_{\min}(X_v) \ge \lambda_{\min}$. Let M denote the random-walk matrix on the 1-skeleton of X with stationary distribution π. For any mean-zero function $f : X(0) \to \mathbb{R}$,

   $$\langle f, Mf \rangle_\pi = \mathbb{E}_{v\sim\pi}[\langle f, M_v f \rangle_{\pi_v}],$$

   where M_v is the walk on X_v. Using $\lambda_{\min}(M_v) \ge \lambda_{\min}$, we have

   $$\langle f|_v, M_v f|_v \rangle_{\pi_v} \ge \mathbb{E}_{\pi_v}[f|_v]^2 + \lambda_{\min} \cdot (\|f|_v\|^2_{\pi_v,2} - \mathbb{E}_{\pi_v}[f|_v]^2) = \lambda_{\min} \|f|_v\|^2_{\pi_v,2} - (1-\lambda_{\min})(Mf(v))^2.$$

   Averaging over v and rearranging gives

   $$\langle f, Mf \rangle_\pi \ge \frac{\lambda_{\min}}{1-\lambda_{\min}} \langle f, f \rangle_\pi.$$

   Hence

   $$\lambda_{\min}(X^{\le 1}) \ge \frac{\lambda_{\min}}{1-\lambda_{\min}}.$$

2. For a d-dimensional X and $\sigma \in X(i)$, the link X_σ has dimension $d-i-1$. Set a_t = smallest eigenvalue for links of dimension t. From (1), $a_{t+1} \geq \frac{a_t}{1-a_t}$ and $a_1 = -1$, so inductively

$$a_t \geq -\frac{1}{t}.$$

Thus for $i < d-2$,

$$\lambda_{\min}(X_\sigma^{\leq 1}) \geq -\frac{1}{d-i-1}.$$

3. X is a d-dimensional one-sided λ-expander, so for all $\sigma \in X(i)$, $i < d-1$,

$$\lambda_2(X_\sigma^{\leq 1}) \leq \lambda.$$

From (2), $\lambda_{\min}(X_\sigma^{\leq 1}) \geq -\frac{1}{d-i-1}$. For the k-skeleton $X^{\leq k}$ we only need $i \leq k-1$, and the worst lower bound occurs at $i = k-1$. Hence every relevant link has

$$|\lambda|_2\,(X_\sigma^{\leq 1}) \leq \max\left(\lambda, \frac{1}{d-k+1}\right),$$

so the two-sided spectral parameter is

$$\gamma = \max\left(\lambda, \frac{1}{d-k+1}\right).$$

□

Q 2 (Agreement test over the complete complex, 6 pts)**.** In lecture we used the following theorem without giving a proof

Theorem 0.2. Let n be large and k moderately sized. For any collection of local functions $\{f_s : s \to \Sigma\}_{s \in \binom{[n]}{k}}$, consider the following direct product test D:

1. Sample a random subset $t \in \binom{[n]}{k/2}$.
2. Then independently sample two subsets s_1, s_2 of size k conditioned on containing t
3. Accept if and only if the two functions agree on their intersection:

$$f_{s_1}|_{s_1 \cap s_2} \equiv f_{s_2}|_{s_1 \cap s_2}.$$

Then for any collection of local functions $\{f_s : s \to \Sigma\}$ there exists a global function $g : [n] \to \Sigma$ such that

$$\Pr_{s \sim \text{Uniform}(\binom{[n]}{k})}[f_s \not\equiv g|_s] \leq O(1) \cdot \Pr_{(s_1,s_2)\sim D}[f_{s_1}|_{s_1\cap s_2} \not\equiv f_{s_2}|_{s_1\cap s_2}].$$

In this problem we will prove this theorem assuming a key lemma. To statement the lemma, we first define for every subset t of size $k/2$ an intermediate function

$$g_{-t} : [n] \setminus t \to \Sigma,\ \ g_{-t}(i) = \text{Majority}\{f_s(i) \mid s \supset t\}.$$

The key lemma states that

Lemma.

$$\mathbb{E}_{t\in\binom{[n]}{k/2}}\left[\Pr_{\substack{s\supset t,\\ s\in\binom{[n]}{k}}}[f_s|_{s\setminus t}\not\equiv g_{-t}|_{s\setminus t}]\right]\le O(1)\cdot \Pr_{(s_1,s_2)\sim D}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}].$$

Use this lemma to finish the proof of the theorem in the following two steps.

1. (3 pts) Prove that there exist two disjoint size-$k/2$ sets t_1, t_2 such that

$$\Pr_{s\supset t_1}[g_{-t_1}|_{s\setminus t_1}\not\equiv f_s|_{s\setminus t_1}]\le O(1)\cdot \Pr_{(s_1,s_2)\sim D}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}],$$

and

$$\Pr_{s\supset t_1}\left[g_{-t_2}|_{t_1}\not\equiv g_{-s\setminus t_1}|_{t_1}\right]\le O(1)\cdot \Pr_{(s_1,s_2)\sim D}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}],$$

and

$$\Pr_{t\in\binom{[n]\setminus(t_1\cup t_2)}{k/2}}[g_{-t_1}|_t\not\equiv g_{-t_2}|_t]\le O(1)\cdot \Pr_{(s_1,s_2)\sim D}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}],$$

2. (3 pts) Define $g:[n]\to\Sigma$ to be

$$g(i)=\begin{cases} g_{-t_1}(i) & i\notin t_1\\ g_{-t_2}(i) & i\in t_1\end{cases}.$$

Prove that

$$\Pr_s[f_s\not\equiv g|_s]\le O(1)\cdot \Pr_{(s_1,s_2)\sim D}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}].$$

Hint: If $f_s\not\equiv g|_s$, then $\Pr_{t\subset s}[f_s|_t\not\equiv g|_t]\ge\frac{1}{2}$.

Solution

Write

$$\Delta := \Pr_{(s_1,s_2)\sim D}\left[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}\right].$$

By the lemma there is an absolute constant $C>0$ such that

$$\mathbb{E}_{t\in\binom{[n]}{k/2}}\left[\Pr_{s\supset t}\left[f_s|_{s\setminus t}\not\equiv g_{-t}|_{s\setminus t}\right]\right]\le C\Delta. \tag{1}$$

1.Existence of disjoint t_1, t_2 with the three bounds. Define for each $t\in\binom{[n]}{k/2}$ the quantity

$$A(t) := \Pr_{s\supset t}\left[f_s|_{s\setminus t}\not\equiv g_{-t}|_{s\setminus t}\right].$$

By (1) we have $\mathbb{E}_t[A(t)]\le C\Delta$, hence there exists some t_1 with

$$A(t_1)\le C\Delta. \tag{i}$$

Next we show that we can pick t_2 disjoint from t_1 so that the other two probabilities are also $\le O(\Delta)$. Fix this t_1 and consider t_2 drawn uniformly from $\binom{[n]\setminus t_1}{k/2}$. For any fixed $s\supset t_1$ and any $i\in t_1$ define the indicator

$$I_{t_2,s,i} := \mathbf{1}\{g_{-t_2}(i)\ne g_{-(s\setminus t_1)}(i)\}.$$

By the definition of majority, for any two index-sets u, v and any coordinate i,

$$\mathbf{1}\{g_{-u}(i) \neq g_{-v}(i)\} \leq \mathbf{1}\{\text{majority over } s \supset u \text{ differs from } f_{s'}(i) \text{ for some } s' \supset v\}$$

and therefore, for every fixed $s \supset t_1$ and $i \in t_1$,

$$\mathbb{E}_{t_2}[I_{t_2,s,i}] \leq \Pr_{s' \supset t_2}[f_{s'}(i) \neq g_{-t_2}(i)] \; + \Pr_{s' \supset (s \setminus t_1)}[f_{s'}(i) \neq g_{-(s\setminus t_1)}(i)].$$

Averaging over $i \in t_1$, over $s \supset t_1$, and then over t_2 gives

$$\mathbb{E}_{t_2}\Big[\Pr_{s \supset t_1}[g_{-t_2}|_{t_1} \not\equiv g_{-(s\setminus t_1)}|_{t_1}]\Big] \leq A(t_2) + A(s \setminus t_1)$$

and averaging once more using (1) (applied to t_2 and to $s \setminus t_1$) yields

$$\mathbb{E}_{t_2}\Big[\Pr_{s \supset t_1}[g_{-t_2}|_{t_1} \not\equiv g_{-(s\setminus t_1)}|_{t_1}]\Big] \leq 2C\Delta.$$

Hence there exists some t_2 (disjoint from t_1) with

$$\Pr_{s \supset t_1}[g_{-t_2}|_{t_1} \not\equiv g_{-(s\setminus t_1)}|_{t_1}] \leq 2C\Delta. \tag{ii}$$

Finally consider the third quantity

$$B(t_1, t_2) := \Pr_{t \in \binom{[n]\setminus(t_1 \cup t_2)}{k/2}}[g_{-t_1}|_t \not\equiv g_{-t_2}|_t].$$

By the same majority-vs-sample argument as above (compare two majorities by comparing to sample values) and averaging over the choices of sets, one gets

$$\mathbb{E}_{t_2}[B(t_1, t_2)] \leq C'\Delta$$

for some absolute constant C', so there exists a choice of t_2 (still disjoint from t_1) such that

$$B(t_1, t_2) \leq C'\Delta. \tag{iii}$$

Using Markov's inequality we can show that with some blowup in the constants $2C, C'$ from (ii),(iii), there exists some t_2 satisfying both inequalities. Thus we complete the proof.

2. The original problem statement has a bug. So please give yourself full credit for this part. Below we solve a slightly different version of this problem where we pick t_1, t_2 satisfying the following conditions (it's not hard to prove such t_1, t_2 exist).

$$\Pr_{s \supset t_1}[g_{-t_1}|_{s\setminus t_1} \not\equiv f_s|_{s\setminus t_1}] \leq O(\Delta),$$

$$\Pr_{s \supset t_2}[g_{-t_2}|_{s\setminus t_2} \not\equiv f_s|_{s\setminus t_2}] \leq O(\Delta), \text{ and}$$

$$\Pr_{(t,s,s') \sim \mu}[f_{s'}|_{s'\setminus t_1} \not\equiv g_{-t_1}|_{s'\setminus t_1}],$$

where the distribution μ is the equal mixture of two distributions μ_1 and μ_2 defined as follows:

μ_1: sample $(t, s, s') \sim D$ conditioned on that $s' \supset (t \cap t_1) \cup t_2$.

μ_2: sample $(t, s, s') \sim D$ conditioned on that $s' \supset t \cup t_1$.

We shall prove that $\Pr_s[f_s \not\equiv g|_s] \le O(\Delta)$.

Fix a random $s \in \binom{[n]}{k}$. If $f_s \not\equiv g|_s$ then by the hint at least half of the $\binom{k}{k/2}$ choices of $t \subset s$ of size $k/2$ satisfy $f_s|_t \not\equiv g|_t$. Therefore

$$\Pr_s[f_s \not\equiv g|_s] \le 2 \Pr_{(s,t)} [f_s|_t \not\equiv g|_t],$$

where t is uniform among $k/2$-subsets of s. To bound the RHS we rewrite it as:

$$\begin{aligned}
\Pr_{(s,t)} [f_s|_t \not\equiv g|_t] &\le \Pr_{(s,t)} [f_s|_{t\cap t_1} \not\equiv g_{-t_2}|_{t\cap t_1}] + \Pr_{(s,t)} [f_s|_{t\setminus t_1} \not\equiv g_{-t_1}|_{t\setminus t_1}] \\
&\le \Pr_{\substack{(s,t,s')\sim D \\ |s'\supset (t\cap t_1)\cup t_2}} [f_s|_{t\cap t_1} \not\equiv f_{s'}|_{t\cap t_1} \vee f_{s'}|_{t\cap t_1} \not\equiv g_{-t_2}|_{t\cap t_1}] \\
&\quad + \Pr_{\substack{(s,t,s')\sim D \\ |s'\supset t\cup t_1}} [f_s|_{t\setminus t_1} \not\equiv f_{s'}|_{t\setminus t_1} \vee f_{s'}|_{t\setminus t_1} \not\equiv g_{-t_1}|_{t\setminus t_1}] \\
&\le \Pr_{(s,t,s')\sim \mu_1} [f_s|_{s\cap s'} \not\equiv f_{s'}|_{s\cap s'}] + \Pr_{(s,t,s')\sim \mu_1} [f_{s'}|_{s'\setminus t_2} \not\equiv g_{-t_2}|_{s'\setminus t_2}] \\
&\quad + \Pr_{(s,t,s')\sim \mu_2} [f_s|_{s\cap s'} \not\equiv f_{s'}|_{s\cap s'}] + \Pr_{(s,t,s')\sim \mu_2} [f_{s'}|_{s'\setminus t_1} \not\equiv g_{-t_1}|_{s'\setminus t_1}] \\
&\le O(\Delta) + O(\Delta) + O(\Delta) + O(\Delta),
\end{aligned}$$

where the last inequality follows from the conditions satisfied by t_1, t_2. Therefore

$$\Pr_s[f_s \not\equiv g|_s] \le O(\Delta).$$

$\square$