CS590 HW5

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- Q1 (1): 2/2. I gave a different cone construction but I believe the argument is correct.
- Q1 (2): 1/2. My claim that the union $K_M^{(2)} \cup C_\ell$ is a 2-dim expander is false b/c the 1-skeleton is disconnected.
- Q2 (1): 2/2. Matches the sample proof.
- Q2 (2): 3/3. Essentially the same counting / probabilistic argument as the sample solution.
- Q3: 3/3. Matches the sample proof.

Total: 11/12.

Solution to problem 1. (1) Let $Y = (V, E)$ be any disconnected graph and consider the cone X by adding a new vertex v, keeping all edges of Y, adding the star edges $\{v, u\}$ for all $v \in V$, and adding the triangle $\{v, a, b\}$ for every $\{a, b\} \in E$. Then $X(2) = \{\{v, a, b\} : \{a, b\} \in E\}$.

We first claim that X is a 1-dim coboundary expander over $\mathbb{F}_2$. We first show that $Z_1(X) = B_1(X)$. To do so, take any $g \in Z_1$. Define $f \in C_0$ by $f(v) = 0$ and $f(u) = g(\{v, u\})$ for $u \in V$. Then for edges $\{v, u\}$ we have $\delta_0 f(\{v, u\}) = g(\{v, u\})$. For $\{a, b\} \in E$, the only triangle containing $\{a, b\}$ is $\{v, a, b\}$, and $\delta_1 g \equiv 0$ gives $g(\{a, b\}) = g(\{v, a\}) + g(\{v, b\}) = \delta_0 f(\{a, b\})$. Hence $g = \delta_0 f$ and $Z_1 = B$. This concludes part 1: because X is finite and the numerator $\|\delta_1 h\|$ vanishes only on $Z_1 = B_1$, we know $\min_{h \in C_1 \setminus B_1} \|\delta_1 h\| / \|h - B_1\|$ is strictly positive. Thus X is a 1-dim coboundary expander.

Next, we show that X is not a 2-dim γ-expander for $\gamma < 11$. Recall

$$X_u(0) = \{e \in X(1) : u \in e\}, \qquad X_u(1) = \{t \in X(2) : u \in t\}.$$

In our cone X, the map $\{v, a\} \leftrightarrow a$ identifies X_v with the base graph Y, so $X_v(0) \sim V(Y)$, $X_v(1) \sim E(Y)$, and $X_v \sim Y$. We use the same notion of random walks on each link $M^+_{\cdot,0}$ as in lecture. Because $X_v \sim Y$ is disconnected, the non-lazy walk $M^+_{v,0}$ on X_v has a nontrivial eigenvalue of 1. Hence no inequality with $\gamma < 1$ can hold, i.e., X is not a 2-dim γ-expander for $\gamma < 1$.

(2) Let m be large and let $K = K_m^{(2)}$ be the complete 2-dim simplicial complex on m vertices (where all edges and all triangles are present). Then, let C_ℓ be a simple cycle on a fresh set of $\ell \geqslant 3$ vertices with no triangles attached. Set X to be the union of K and C_ℓ. This is a 2-dim simplicial complex.

We first claim that X is a 2-dim expander. In K, every link is the complete graph K_{m-1}. For the non-lazy random walk on a complete graph, all nontrivial eigenvalues are strictly inside $(-1, 1)$. Hence each X_v is a spectral expander, i.e., K is a 2-dim γ-expander for some $\gamma < 1$. The disjoint cycle C_ℓ has no triangles so it does not affect the previous argument. *Update: actually I think my solution for this part is incorrect (X is not a 2-dim expander) and don't see how to fix it.*

We now show that X is not a 1-dim β-coboundary expander for any $\beta > 0$. Recall with $B_1 = \mathrm{im}\delta_0$ (edge cuts) and $Z_1 = \ker \delta_1$ (every triangle sees an even number of selected egdes), X would be a 1-dim β-coboundary expander iff $B_1 = Z_1$ and

$$\min_{f \notin B_1} \|\delta_1 f\| / \|f - B_1\| \geqslant \beta,$$

in particular, bounded away from 0. Look at the C_ℓ component. It has no triangles so $\delta_1 \equiv 0$ there. Take $f = C_1(X)$ to be the indicator of the ℓ edges of the cycle C_ℓ. Then:

- $f \in Z$ vacuously, because there are no 2-faces on that component, but
- $f \notin B_1$, for on a cycle, B_1 is a cut, while the whole cycle is not a cut.

Hence $\|\delta_1 f\| = 0$ while $\|f - B_1\| > 0$. This completes the proof.

Solution to problem 2. Let X be the 2-dim complete complex on n vertices (n used in part (2)).

- For any edge $\{u, w\} \in X(1)$,

$$\delta_0 g_v(\{u, w\}) = g_v(u) + g_v(w) = f(\{u, v\}) + f(\{v, w\}).$$

  Also, for the triangle $\{u, v, w\} \in X(2)$,

$$\delta_1 f(\{u, v, w\}) = f(\{u, v\}) + f(\{v, w\}) + f(\{u, w\}).$$

  Therefore, if $f(\{u, w\}) \neq \delta_0 g_v(\{u, w\})$ then over $\mathbb{F}_2$,

$$\delta_1 f(\{u, v, w\}) = \delta_0 g_v(\{u, w\}) + f(\{u, w\}) = 1 \neq 0.$$

- Because each $\delta_0 g_v \in B_1$, we have

$$\|f - B_1\| = \min_{b \in B_1} \|f - b\| \leqslant \mathbb{E}_{v \in \pi(0)} \|f - \delta_0 g_v\|,$$

  i.e., the hint. Now fix v. For any edge $\{u, w\}$ *not* incident to v, $f(\{u, w\}) \neq \delta_0 g_v(\{u, w\})$ if and only if $\delta_1 f(\{u, v, w\}) \neq 0$ by the previous part. Therefore, for this fixed v, the set of disagreements between f and $\delta_0 g_v$ is in the bijection with the set of triangles containing v on which $\delta_1 f$ is nonzero. Consequently,

$$\|f - \delta_0 g_v\| = \frac{|\{\text{bad triangles containing } v\}|}{|X(1)|}.$$

  Averaging over v, it is easy to see that each bad triangle is counded precisely 3 times, once per vertex. Hence

$$\mathbb{E}_v \|f - \delta_0 g_v\| = \frac{3}{n|X(1)|} \cdot |\{\text{bad triangles}\}| = \frac{3|X(2)|}{n|X(1)|} \cdot \|\delta_1 f\|.$$

  As X is the complete complex on 2-vertices, $|X(1)| = \binom{n}{2}$ and $|X(2)| = \binom{n}{3}$, so $3|X(2)|/(n|X(1)|) = (n-2)/n$. Plugging this into the above identity, we obtain $\mathbb{E}_v \|f - \delta_0 g_v\| = (n-2)/n \cdot \|\delta_1 f\|$, i.e.,

$$\frac{\|\delta_1 f\|}{\|f - B_1\|} \geqslant \frac{n}{n-2} \geqslant 1,$$

  i.e., X is a 1-dim 1-coboundary expander for all n.

Solution to problem 3. For each row a, choose $w_a \in C_B$ minimizing the row distance to $f(a,\cdot)$; likewise, for each column b, choose $w_b \in C_A$ minimizing the column distance to $f(\cdot,b)$. Then

$$\begin{aligned}\mathrm{dist}_{\mathrm{row}}(f) &= \mathbb{E}_b[\mathrm{dist}(f(\cdot,b),C_A)] = \mathbb{E}_{a,b}[\mathbf{1}\{w_b(a) \neq f(a,b)\}],\\ \mathrm{dist}_{\mathrm{col}}(f) &= \mathbb{E}_a[\mathrm{dist}(f(a,\cdot),C_B)] = \mathbb{E}_{a,b}[\mathbf{1}\{w_a(b) \neq f(a,b)\}].\end{aligned}$$

For every (a,b), since $\{w_a(b) \neq w_b(a)\} \subset \{w_a(b) \neq f(a,b)\} \cup \{f(a,b) \neq w_b(a)\}$, the row/column disagreement is controlled by $d(f)$:

$$\mathbb{P}_{a,b}[w_a(b) \neq w_b(a)] \leqslant \mathrm{dist}_{\mathrm{col}}(f) + \mathrm{dist}_{\mathrm{row}}(f) = 2d(f).$$

Invoking the β-agreement testability of $C_A \otimes C_B$, we see that there exists a global $c \in C_A \otimes C_B$ with

$$\beta \cdot \Big(\frac{1}{2}\mathbb{P}_a[c(a,\cdot) \neq w_a] + \frac{1}{2}\mathbb{P}_b[c(\cdot,b) \neq w_b]\Big) \leqslant 2d(f).$$

Therefore the quantity enclosed in the parenthesis on the LHS, call it A, is at most $2/\beta \cdot d(f)$.
Finally we relate c back to f. We use two triangle inequalities: $f(a,b)-w_a(b)-c(a,b)$, and $f(a,b)-w_b(a)-c(a,b)$:

$$\begin{aligned}2 \cdot \mathbf{1}\{f(a,b) \neq c(a,b)\} \leqslant{}& \mathbf{1}\{w_a(b) \neq c(a,b)\} + \mathbf{1}\{w_b(a) \neq c(a,b)\}\\ &+ \mathbf{1}\{w_a(b) \neq f(a,b)\} + \mathbf{1}\{w_b(a) \neq f(a,b)\}.\end{aligned}$$

Averagging, $\mathrm{dist}(f,c) \leqslant A + d(f) \leqslant (2/\beta + 1)d(f) = (\beta+2)/\beta \cdot d(f)$. Finally, since $c \in C_A \otimes C_B$,

$$\mathrm{dist}(f, C_A \otimes C_B) \leqslant \mathrm{dist}(f,c) \leqslant \frac{\beta+2}{\beta}d(f),$$

and the proof is complete.