Homework 5

Release date: Nov. 9, Due date: Nov. 16.

Guidelines: Submit your solutions in pdf format on Gradescope by **5pm on Nov. 16**. Please be sure to tag each of your solutions to the correct problem. You are encouraged to form small groups to work through the homework, but you **must** write up all your solutions on your own. Homework in this class will be self-graded. To help keep things fair, a few assignments will be randomly checked to make sure the grading is accurate.

Q 1 (High dimensional expansion and coboundary expansion, 4 pts)**.** Give constructions of 2-dimensional simplicial complexes that satisfy each of the following requirements:

1. (2 pts) A 2-dimensional simplicial complex that is a 1-dimensional coboundary expander (over $\mathbb{F}_2$) but is not a 2-dimensional γ-expander for any $\gamma < 1$.
2. (2 pts) A 2-dimensional simplicial complex that is a 2-dimensional expander but is not a 1-dimensional β-coboundary expander (over $\mathbb{F}_2$) for any constant $\beta > 0$.

Q 2 (Coboundary expansion of complete complexes, 5 pts)**.** In this question we will show that the 2-dimensional complete complex is a 1-dimensional 1-coboundary expander.

This problem guides you towards a proof.

1. (2 pts) For every $v \in X(0)$, define the function $g_v : X(0) \to \mathbb{F}_2$ as follows:
$$g_v(v) = 0, \; g_v(u) = f(\{u, v\}).$$
Prove that if for some $\{u, w\} \in X(1)$ it holds that $f(\{u, w\}) \neq \delta_0 g_v(\{u, w\})$, then $\delta_1 f(\{u, v, w\}) \neq 0$.
2. (3 pts) Finish the proof that X is a 1-dimensional 1-coboundary expander.
Hint: First show that $\|f - B^1\| \leq \mathbb{E}_{v \sim \pi(0)}[\|f - \delta_0 g_v\|]$.

Q 3 (Agreement testability implies robust testability, 3 pts)**.** Recall that in class we defined β-agreement testability of tensor codes. Now we define the closely related notation of ρ-robust testability of tensor codes.

Definition. Consider a family of tensor codes $C_A \otimes C_B \in \mathbb{F}^{\Delta \times \Delta}$. For any $f \in \mathbb{F}^{\Delta \times \Delta}$, let

$$\mathrm{dist}_{col}(f) = \mathrm{dist}(f, \mathbb{F}^{\Delta} \otimes C_B), \; \mathrm{dist}_{row}(f) = \mathrm{dist}(f, C_A \otimes \mathbb{F}^{\Delta}),$$

and

$$d(f) = \frac{\mathrm{dist}_{col}(f) + \mathrm{dist}_{row}(f)}{2}.$$

$C_A \otimes C_B$ is ρ-robust testable if

$$\rho \cdot \mathrm{dist}(f, C_A \otimes C_B) \leq d(f).$$

Prove that if $C_A \otimes C_B$ is β-agreement testable, then it is also $\frac{\beta}{\beta+2}$-robust testable.