590.05 Expanders, High-dimensional Expanders, and their Applications Fall 2025

Homework 5 Solution

Release date: Nov. 16, Due date: Nov. 23.

Guidelines: Submit your self-grade in pdf format on Gradescope by **5pm on Nov. 23**. Please compare your solutions with the posted ones and assign yourself points honestly:

- Full credit: essentially correct
- 50% credit: good progress but errors/incomplete
- 25% credit: some relevant ideas only
- No credit: no real attempt

If you take partial credit, briefly note why.

Q 1 (High dimensional expansion and coboundary expansion, 4 pts)**.** Give constructions of 2-dimensional simplicial complexes that satisfy each of the following requirements:

1. (2 pts) A 2-dimensional simplicial complex that is a 1-dimensional coboundary expander (over $\mathbb{F}_2$) but is not a 2-dimensional γ-expander for any $\gamma < 1$.
2. (2 pts) A 2-dimensional simplicial complex that is a 2-dimensional expander but is not a 1-dimensional β-coboundary expander (over $\mathbb{F}_2$) for any constant $\beta > 0$.

Solution

1. One simple example is obtained by taking the disjoint union of two coboundary expanders. It is easy to verify that the resulting complex is still a coboundary expander with slightly worse parameters. However, because the 1-skeleton of the new complex is disconnected, it fails to be a 2-dimensional γ-expander for any $\gamma < 1$.

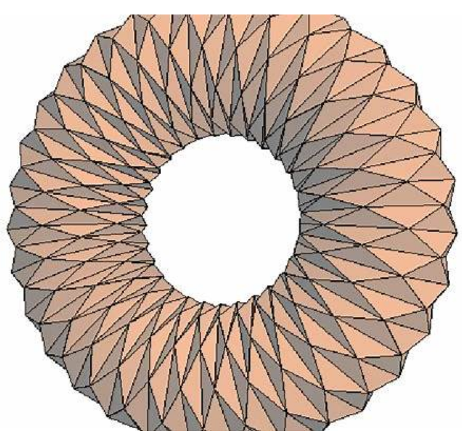

Figure 1: A tessellation of a torus by triangles, where each vertex is the meeting point of six triangles.

2. One such example is illustrated above: a 2-dimensional complex arising from a triangular tessellation of the torus. Since the complex contains 1-dimensional holes, we have $Z^1 \neq B^1$, and hence it is not a 1-dimensional coboundary expander. Nevertheless, the 1-skeleton has second eigenvalue less than 1, and each vertex link is a 6-cycle with second eigenvalue $\frac{1}{2}$. Therefore, the complex is a 2-dimensional expander.

□

Q 2 (Coboundary expansion of complete complexes, 5 pts)**.** In this question we will show that the 2-dimensional complete complex is a 1-dimensional 1-coboundary expander.

This problem guides you towards a proof.

1. (2 pts) For every $v \in X(0)$, define the function $g_v : X(0) \to \mathbb{F}_2$ as follows:

$$g_v(v) = 0, \; g_v(u) = f(\{u, v\}).$$

Prove that if for some $\{u, w\} \in X(1)$ it holds that $f(\{u, w\}) \neq \delta_0 g_v(\{u, w\})$, then $\delta_1 f(\{u, v, w\}) \neq 0$.

2. (3 pts) Finish the proof that X is a 1-dimensional 1-coboundary expander.

Hint: First show that $\|f - B^1\| \leq \mathbb{E}_{v \sim \pi(0)}[\|f - \delta_0 g_v\|]$.

Solution

1. Under this condition we have that

$$\delta_1 f(\{u, v, w\}) = f(\{u, v\}) + f(\{v, w\}) + f(\{u, w\}) = \delta_0 g_v(\{u, w\}) + f(\{u, w\}) = 1,$$

where the last equality comes from $\delta_0 g_v(\{u, w\}) \neq f(\{u, w\})$.

2. Since for every $v \in X(0)$,

$$\left\| f - B^1 \right\| = \min_{h \in B^1} \|f - h\| \leq \|f - \delta_0 g_v\| .$$

Therefore taking the average over $v \sim \pi(0)$ we have

$$\begin{aligned} \left\| f - B^1 \right\| &\leq \mathbb{E}_{v \sim \pi(0)} \left[\|f - \delta_0 g_v\| \right] \\ &= \mathbb{E}_{v \sim \pi(0)} \left[\Pr_{\{u,w\} \sim \pi(1)} [f(\{u, w\}) \neq \delta_0 g_v(\{u, w\})] \right] \\ &\leq \mathbb{E}_{v \sim \pi(0)} \left[\Pr_{\{u,w\} \sim \pi_v(1)} [\delta_1 f(\{u, v, w\}) \neq 0] \right] \\ &= \mathbb{E}_{\{u,v,w\} \sim \pi(2)} \left[\delta_1 f(\{u, v, w\}) \neq 0 \right] = \|\delta_1 f\| . \end{aligned}$$

Therefore the complete complex X is a 1-dimensional coboundary expander.

□

Q 3 (Agreement testability implies robust testability, 3 pts)**.** Recall that in class we defined β-agreement testability of tensor codes. Now we define the closely related notation of ρ-robust testability of tensor codes.

Definition. Consider a family of tensor codes $C_A \otimes C_B \in \mathbb{F}^{\Delta\times\Delta}$. For any $f \in \mathbb{F}^{\Delta\times\Delta}$, let

$$\mathrm{dist}_{col}(f) = \mathrm{dist}(f, \mathbb{F}^{\Delta} \otimes C_B),\ \mathrm{dist}_{row}(f) = \mathrm{dist}(f, C_A \otimes \mathbb{F}^{\Delta}),$$

and

$$d(f) = \frac{\mathrm{dist}_{col}(f) + \mathrm{dist}_{row}(f)}{2}.$$

$C_A \otimes C_B$ is ρ-robust testable if

$$\rho \cdot \mathrm{dist}(f, C_A \otimes C_B) \le d(f).$$

Prove that if $C_A \otimes C_B$ is β-agreement testable, then it is also $\frac{\beta}{\beta+2}$-robust testable.

Solution

Given any $f \in \mathbb{F}^{\Delta\times\Delta}$, for each row index a, pick a string $w_a \in C_B$ minimizing the distance to the row $f(a,\cdot)$, and for each column index b, pick a string $w_b \in C_A$ minimizing the distance to the column $f(\cdot,b)$.

Then

$$\mathrm{dist}_{col}(f) = \Pr_{a,b}[w_a(b) \neq f(a,b)], \quad \mathrm{dist}_{row}(f) = \Pr_{a,b}[w_b(a) \neq f(a,b)],$$

and

$$d(f) = \frac{\mathrm{dist}_{col}(f) + \mathrm{dist}_{row}(f)}{2}.$$

Step 1: Relating disagreement probaility to $d(f)$. For each (a,b),

$$\mathbf{1}\{w_a(b) \neq w_b(a)\} \le \mathbf{1}\{w_a(b) \neq f(a,b)\} + \mathbf{1}\{w_b(a) \neq f(a,b)\}.$$

Averaging over (a,b) gives

$$\Pr_{a,b}[w_a(b) \neq w_b(a)] \le \Pr_{a,b}[w_a(b) \neq f(a,b)] + \Pr_{a,b}[w_b(a) \neq f(a,b)] = 2d(f).$$

Step 2: Bounding the distance to a global codeword. For any $c \in C_A \otimes C_B$, by the triangle inequality,

$$\Pr_{a,b}[c(a,b) \neq f(a,b)] \le \Pr_{a,b}[c(a,b) \neq w_a(b)] + \Pr_{a,b}[w_a(b) \neq f(a,b)].$$

Averaging the analogous inequality over both rows and columns yields

$$\mathrm{dist}(f,c) \le \frac{1}{2}\Big(\Pr_a[c(a,\cdot) \not\equiv w_a] + \Pr_b[c(\cdot,b) \not\equiv w_b]\Big) + d(f).$$

Step 3: Apply β-agreement testability. By β-agreement testability, there exists $c \in C_A \otimes C_B$ such that

$$\frac{\beta}{2}\Big(\Pr_a[c(a,\cdot) \neq w_a] + \Pr_b[c(\cdot,b) \neq w_b]\Big) \le \Pr_{a,b}[w_a(b) \neq w_b(a)].$$

Using Step 1, we have

$$\frac{1}{2}\Big(\Pr_a[c(a,\cdot) \neq w_a] + \Pr_b[c(\cdot,b) \neq w_b]\Big) \le \frac{1}{\beta}\Pr_{a,b}[w_a(b) \neq w_b(a)] \le \frac{2}{\beta}d(f).$$

Plugging this into the bound from Step 2 gives

$$\mathrm{dist}(f,c) \le \frac{2}{\beta}d(f) + d(f) = \frac{\beta+2}{\beta}\,d(f).$$

Since $c \in C_A \otimes C_B$, it follows that

$$\frac{\beta}{\beta+2} \cdot \operatorname{dist}(f, C_A \otimes C_B) \leq d(f).$$

Thus $C_A \otimes C_B$ is ρ-robust testable with

$$\rho = \frac{\beta}{\beta+2}.$$

$\square$