590.05 Expanders and High-dimensional Expanders **Fall 2025**

Lecture 1: Introduction to Expanders

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1.1 Motivations of Expanders

Informally, expanders are graphs without very sparse cuts. The formal definition is given below.

Definition 1.1. *An undirected graph $G = (V, E)$ is a one-sided λ-expander if its normalized adjacency matrix $\overline{A}$ satisfies that $\lambda_2(\overline{A}) \leq \lambda$. G is a two-sided λ-expander if $|\lambda_2|(\overline{A}) \leq \lambda$ where $|\lambda_2|(\overline{A})$ is the second largest eigenvalue of $\overline{A}$ in absolute value.*

Despite the simple definition, expanders are really useful objects in various areas in computer science, partially because the spectral condition implies several nice structural properties in the graph. We shall examine a couple of such properties later. Now to demonstrate the range of different problems that can be solved with using , we give a couple examples here. After the first two weeks of the lectures you will be able to solve them yourself.

Problem 1.2 (How to draw graphs in 2D?). *Given a graph $G = (V, E)$ how do we draw the graph in $\mathbb{R}^2$ nicely? Here by nice, we would want vertices that are adjacent in the graph to be close to each other in Eculidean distance. More formally we phrase the requirement to be that find a map from $M : V \to \mathbb{R}^2$ such that $\sum_{v \in V} \|M(v)\|_2^2 = 1$ and the sum of squared distances $\sum_{uv \in E} \|M(u) - M(v)\|_2^2$ is minimized.*

Problem 1.3 (The sparest cut problem). *In a graph $G = (V, E)$, a cut induced by $S \subseteq V$ is the collection of edges (denoted by ∂S, the "boundary" of S) between S and its complement. A cut induced by S ($|S| \leq |V|/2$) is sparse if only a small fraction of edges adjacent to S are in the cut ∂S. The computational problem is given a graph G can we efficiently find the sparsest cut in G? Note that the exact problem is* NP*-hard, so we just need a good approximation algorithm.*

Problem 1.4 (Good linear Codes). *A linear code C_n is a subspace in some vector space ${\mathbb{F}_p}^n$. A main application of codes is data transmission across noisy channels. A couple relevant parameters include the rate of the code $r(C_n) = \frac{\dim(C_n)}{n}$, the distance of the code $\delta(C_n) = \min_{c \in C_n \setminus \vec{0}} \mathrm{Hamming}(c)$, and the running time $t(C_n)$ of the decoding algorithm that takes in a corrupted codeword and outputs the codeword closest to the input.*

The question is: can we construct an infinite family of linear codes $\{C_n\}_n$ such that there exists $r_0, \delta_0 > 0$ and $T(n) \in O(n)$ satisfying that for all C_n, $r(C_n) \geq r_0$, $\delta(C_n) \geq \delta_0$, and C_n has a decoding algorithm running in time $T(n)$?

Problem 1.5. *Derandomizing* BPP *Recall that* BPP *is the class of languages that can be solved by probabilistic polynomial time Turing machines with bounded, two-sided errors. The common belief is that* BPP = P, *so an important question is to understand how much randomness is actually needed for these problems. While proving the holy grail that $O(\log n)$ bits of randomness suffices, there have been many successes in showing that it is possible to reduce the amount randomness. Here is a question along this line.*

Given an algorithm A for a BPP *language L that uses m random bits and achieves $< \frac{1}{3}$ two-sided errors, one can reduce the errors to any $\varepsilon > 0$ by repeating the algorithm $\log(1/\varepsilon)$ times and output the majority vote of the outputs. However this approach uses $m \log(1/\varepsilon)$ random bits. Can we achieve the same error bound while using only $m + O(\log(1/\varepsilon))$ bits of randomness?*

The usefulness of expanders root in the fact that the spectral definition surprisingly implies an array of combinatorial, geometric, and probabilistic properties. We next carefully examine these properties.

1.2 Notations and Facts

In this section we introduce notations and facts about graphs and matrix spectra.

Give a graph $G = (V, E)$ on n vertices, we use D to denote the diagonal matrix of vertex degrees, A to denote the adjacency matrix, and L to denote the unnormalized graph Laplacian $D - A$.

We also define the normalized adjacency matrix to be $\overline{A} = D^{-1/2}AD^{-1/2}$, and the normalized graph Laplacian to be $\overline{L} = D^{-1/2}LD^{-1/2} = I - \overline{A}$.

Let the spectrum of $\overline{A}$ be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and the corresponding eigenvectors be $v_1, v_2 \ldots, v_n$. Let the spectrum of L be $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and the corresponding eigenvectors be $u_1, u_2 \ldots, u_n$. Here are some facts about the spectra and eigenspaces of $\overline{A}$ and $\overline{L}$.

Fact 1.6. *1. $\lambda_1 = 1$ and $v_1 \propto D^{1/2}\vec{1}$, $\lambda_n \geq -1$. 2. for all $i \in [n]$, $\gamma_i = 1 - \lambda_i$ and $u_i = v_i$.*

Fact 1.7. *For a matrix M with spectrum $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and corresponding eigenvectors $u_1, u_2, \ldots, u_n$, its i-th smallest eigenvalue can be equivalently defined using Raleigh quotient as*

$$\lambda_i = \min_{x \perp \mathrm{Span}(u_1,\ldots,u_{i-1})} \frac{x^\top Mx}{\langle x, x\rangle}.$$

Fact 1.8. *For an unweighted graph G, the quadratic forms of L and $\overline{L}$ can be written as*

$$\langle f, Lf\rangle = \sum_{ab\in E} (f(a) - f(b))^2, \ \langle f, \overline{L}f\rangle = \sum_{ab\in E} \left(\frac{1}{\sqrt{d_a}}f(a) - \frac{1}{\sqrt{d_b}}f(b)\right)^2.$$

where d_a, d_b are the degrees of a and b inside G.

1.3 Geometric property: conductance

The first implication of graph expansion is that for every set $S \subset V$ that is not too large at least $\frac{\gamma_2}{2}$-fraction of its adjacent edges are in the cut ∂S.

Definition 1.9 (Conductance)**.** *Given a graph $G = (V, E)$ and a set $S \subseteq V$, we define the conductance of the set to be*

$$\phi(S) = \frac{|\partial S|}{\min(\mathrm{vol}(S), \mathrm{vol}(\overline{S}))},$$

where $\mathrm{vol}(S) = \sum_{a\in S} d_a$ *and* $\overline{S} = V \setminus S$.

Just like the original physics notion that measures how easily an electric current flows through a material, here conductance measures how easily a random walk in G starting from a set S escapes S.

Definition 1.10 (Graph conductance)**.** *The conductance of a graph $G = (V, E)$ is*

$$\phi(G) = \min_{S\subseteq V} \phi(S).$$

The following classic result captures the connections between graph expansion and conductance.

Lemma 1.11. *(Cheeger's inequality) Let G be a graph with Laplacian spectrum $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$. Then*

$$\frac{\gamma_2}{2} \leq \phi(G) \leq \sqrt{2\gamma_2}.$$

One consequence of the lower bound on $\phi(G)$ is that if γ_2 is bounded away from 0, then it is hard to disconnect G. In particular, even after removing ε-fraction of the edges, G still has a large connected component of size $\geq (1-\varepsilon/2\phi(G))n$. The proof is left as an exercise.

We first prove the lower bound, and postpone the proof of the upper bound till later.

Proof of the lower bound side of Cheeger's inequality. Let $x = D^{1/2}\vec{1}$ denote the smallest eigenvector L of G.

For any $S \subseteq V$ such that $\mathrm{vol}(S) \leq \mathrm{vol}(\overline{S})$, let $\vec{1}_S$ be the indicator vector of S and

$$f = D^{1/2}\vec{1}_S - \left\langle \vec{1}_S, D\vec{1} \right\rangle \cdot x / \|x\|^2 .$$

Note that $\langle f, x\rangle = \left\langle \vec{1}_S, D\vec{1} \right\rangle - \left\langle \vec{1}_S, D\vec{1} \right\rangle = 0$ so $f \perp x$. Compute

$$\langle f, f\rangle = \left\langle \vec{1}_S, D\vec{1}_S \right\rangle + \left\langle \vec{1}_S, D\vec{1} \right\rangle^2 / \|x\|^2 - 2\left\langle \vec{1}_S, D\vec{1} \right\rangle^2 / \|x\|^2 = \mathrm{vol}(S) - \mathrm{vol}(S)^2/\mathrm{vol}(V),$$

and

$$f^\top L f = \left\langle \vec{1}_S, (D-A)\vec{1}_S \right\rangle = \mathrm{vol}(S) - 2\left|E(S,S)\right| = \left|E(S,\overline{S})\right|$$

Recall that by Fact 1.7 every $f \perp x$ satisfies that $\gamma_2 \leq \frac{z^\top A z}{\langle z, z\rangle}$.

$$\gamma_2 \leq \frac{f^\top L f}{\langle f, f\rangle} = \frac{\left|E(S,\overline{S})\right|}{\mathrm{vol}(S) - \mathrm{vol}(S)^2/\mathrm{vol}(V)} \leq \frac{\left|E(S,\overline{S})\right|}{\mathrm{vol}(S)/2} = 2\phi(S).$$

Here the second inequality follows from the condition that $\mathrm{vol}(S) \leq \mathrm{vol}(\overline{S})$. Since this holds for every set S satisfying this condition, we can conclude that

$$\phi(G) = \min_{S \mid \mathrm{vol}(S) \leq \mathrm{vol}(\overline{S})} \phi(S) \geq \frac{\gamma_2}{2}.$$

□