

## Lecture 6: An Explicit Construction of Expanders

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## 6.1 Random constructions

Though the proof that random  $d$ -regular graphs are Ramanujan with constant probability is quite advanced, one can still use elementary techniques to show that random  $d$ -regular graphs are one-sided  $\frac{31}{32}$ -expanders with high probability. This follows from a lower bound on the conductance of random  $d$ -regular graphs, obtained via a union bound argument. We state the theorem and leave its proof as an exercise.

**Theorem 6.1.** *For every fixed  $d \geq 3$ , a uniformly random  $d$ -regular graph on  $n$  vertices is a one-sided  $\frac{31}{32}$ -expander with probability  $1 - o(1)$ .*

## 6.2 Margulis–Gabber–Galil expanders

We now move to the first explicit construction of expanders due to Margulis that was later analyzed by Gabber and Galil.

**Definition 6.2** (Margulis–Gabber–Galil Graph). *Let  $n \in \mathbb{N}$ . The Margulis–Gabber–Galil graph  $G_n = (V, E)$  is defined as follows:*

$$V = \mathbb{Z}_n \times \mathbb{Z}_n,$$

and each vertex  $(x, y) \in V$  is connected to the following 8 neighbors (all coordinates modulo  $n$ ):

$$(x, y) \mapsto (x \pm y, y), (x, y) \mapsto (x \pm 1, y), (x, y) \mapsto (x, y \pm x), (x, y) \mapsto (x, y \pm 1).$$

Thus  $G_n$  is an 8-regular graph on  $n^2$  vertices.

In this section we prove that  $G_n$  is an expander:

**Theorem 6.3** (Gabber–Galil). *The Margulis–Gabber–Galil graph has spectral gap (second smallest eigenvalue of the normalized Laplacian)  $\gamma(G_n) \geq \frac{1}{600}$ .*

To present the proof we need to define two more graphs  $R_n$  and  $Z$ .

**Definition 6.4.** *Let  $R_n$  be the uncountable graph with vertex set  $V = [0, n)^2$  and edges of the form  $(x, y) \mapsto (x \pm y, y), (x, y) \mapsto (x, y \pm x)$ .*

Though this is a graph with uncountable number of vertices, we can still define functions over the graph, Laplacian operators, inner product, and spectral gap.

For any function  $f, g : V \rightarrow \mathbb{R}$ , define the inner product be

$$\langle f, g \rangle = \int_{[0,n]^2} f(x, y) \cdot g(x, y) dx dy$$

Laplacian operator be

$$\langle f, \bar{L}_R g \rangle = \frac{1}{4} \int_{[0,n]^2} (f(x, y) - g(x, x+y))^2 + (f(x, y) - g(x+y, y))^2 dx dy,$$

and the spectral gap be

$$\gamma(R_n) = \inf_{f \perp \langle f, \mathbb{1} \rangle = 0} \frac{\langle f, \bar{L}_R f \rangle}{\langle f, f \rangle}.$$

The next graph we will consider is also an infinite graph.

**Definition 6.5.** Let  $Z$  be the uncountable graph with vertex set  $V = \mathbb{Z}^2 \setminus \{(0, 0)\}$  and edges of the form  $(x, y) \rightarrow (x \pm y, y), (x, y) \rightarrow (x, y \pm x)$ .

Unlike  $R_n$ ,  $Z$  is a discrete graph we can define normalized Laplacian operator just as before. The only difference is that instead of considering all function over  $V$ , we only focus on functions with bounded norm. As a result the spectral gap is defined as follows

$$\gamma(Z) = \inf_{f \mid \|f\|_2 < \infty} \frac{\langle f, \bar{L}_Z f \rangle}{\langle f, f \rangle}.$$

Now to prove the theorem it boils down to connect the spectral gaps of the three graphs using the following lemmas.

**Lemma 6.6.**

$$\gamma(G_n) \geq \frac{1}{3} \gamma(R_n).$$

This first lemma we will skip the proof and leave it as an exercise.

**Lemma 6.7.**

$$\gamma(R_n) \geq \gamma(Z).$$

**Lemma 6.8.**

$$\phi(Z) \geq \frac{1}{10}.$$

Now we move on to prove these two lemmas. To prove Lemma 6.7 we recall some basic results from Fourier analysis.

### 6.2.1 Fourier analysis of bivariate polynomials

Let  $n \in \mathbb{N}$  and consider functions  $f : [0, n]^2 \rightarrow \mathbb{R}$ . The Fourier characters on  $\mathbb{Z}^2$  are

$$\chi_{k,\ell}(x, y) = \frac{1}{n} \exp(2\pi i(kx + \ell y)), \quad (k, \ell) \in \mathbb{Z}^2.$$

They form an orthonormal basis with respect to the inner product

$$\langle f, g \rangle = \int_{[0, n]^2} f(x, y) \cdot g(x, y) dx dy$$

The Fourier coefficients of  $f$  are defined as

$$\widehat{f}(k, \ell) = \int_{[0, n]^2} f(x, y) \cdot \chi_{k, \ell}(x, y) dx dy.$$

The Fourier expansion of  $f$  is then

$$f(x, y) = \sum_{k, \ell \in \mathbb{Z}^2} \widehat{f}(k, \ell) \chi_{k, \ell}(x, y).$$

Then the Parseval identity gives that for any  $f : [0, n]^2 \rightarrow \mathbb{R}$ ,

$$\langle f, f \rangle = \sum_{k, \ell \in \mathbb{Z}^2} \widehat{f}(k, \ell)^2.$$

### 6.2.2 Proof of Lemma 6.7

The main idea is to show that for any  $f : [0, n]^2 \rightarrow \mathbb{R}$  with bounded norm, let  $\widehat{f} : \mathbb{Z}^2 \rightarrow \mathbb{R}$  be its Fourier coefficient function, then the two functions satisfy

$$\frac{\langle f, \bar{L}_R f \rangle}{\langle f, f \rangle} = \frac{\langle \widehat{f}, \bar{L}_Z \widehat{f} \rangle}{\langle \widehat{f}, \widehat{f} \rangle}.$$

From the Parseval identity its clear that the two denominators are equal, we now establish the equality between the two numerators.

Define  $s(x, y) = f(x, y) - f(x, x + y)$  and  $t(x, y) = f(x, y) - f(x + y, y)$ , then

$$\begin{aligned} \langle f, \bar{L}_R f \rangle &= \frac{1}{4} \int_{[0, n]^2} (f(x, y) - f(x, x + y))^2 + (f(x, y) - f(x + y, y))^2 dx dy \\ &= \frac{1}{4} \int_{[0, n]^2} s(x, y)^2 + t(x, y)^2 dx dy \\ &= \frac{1}{4} \sum_{k, \ell \in \mathbb{Z}^2} \widehat{s}(k, \ell)^2 + \widehat{t}(k, \ell)^2. \end{aligned}$$

Furthermore, by Fourier transform

$$\begin{aligned} \widehat{s}(k, \ell) &= \int_{[0, n]^2} (f(x, y) - f(x, x + y)) \chi_{k, \ell}(x, y) dx dy \\ &= \widehat{f}(k, \ell) - \frac{1}{n^2} \int_{[0, n]^2} f(x, x + y) \cdot \exp(2\pi i(kx + \ell y)) dx dy \\ &= \widehat{f}(k, \ell) - \frac{1}{n^2} \int_{[0, n]^2} f(x, x + y) \cdot \exp(2\pi i((k - \ell)x + \ell(x + y))) dx dy \\ &= \widehat{f}(k, \ell) - \widehat{f}(k - \ell, \ell). \end{aligned}$$

Similarly we can deduce that  $\widehat{t}(k, \ell) = \widehat{f}(k, \ell) - \widehat{f}(k, \ell - k)$ . Plugging in the two values, we get that

$$\begin{aligned} \langle f, \bar{L}_R f \rangle &= \frac{1}{4} \sum_{k, \ell \in \mathbb{Z}^2} \left( \widehat{f}(k, \ell) - \widehat{f}(k - \ell, \ell) \right)^2 + \left( \widehat{f}(k, \ell) - \widehat{f}(k, \ell - k) \right)^2 \\ &= \langle \widehat{f}, \bar{L}_Z \widehat{f} \rangle \end{aligned}$$

Thus we complete the proof.

### 6.2.3 Proof of Lemma 6.8

For any subset  $A \subseteq \mathbb{Z}^2 \setminus \{(0, 0)\}$ , we partition  $A$  into 5 parts,  $A_0 \sqcup A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4$  where  $A_0 = A \cap (x\text{-axis} \cup y\text{-axis})$ , and  $A_i = A \cap i\text{-th quadrant}$  for  $i \geq 1$ .

For every set  $A_i$  define

$$\begin{aligned} S(A_i) &= \{(x + y, y) \mid (x, y) \in A_i\}, & S^{-1}(A_i) &= \{(x - y, y) \mid (x, y) \in A_i\} \\ T(A_i) &= \{(x, y + x) \mid (x, y) \in A_i\}, & T^{-1}(A_i) &= \{(x, y - x) \mid (x, y) \in A_i\}. \end{aligned}$$

We observe that (1)

$$|S(A_i)| = |S^{-1}(A_i)| = |A_i| = |T(A_i)| = |T^{-1}(A_i)|,$$

and (2)  $S(A_1), T(A_1)$  are in the 1-st quadrant,  $S^{-1}(A_2), T^{-1}(A_2)$  are in the 2-nd quadrant,  $S(A_3), T(A_3)$  are in the 3-rd quadrant, and  $S^{-1}(A_4), T^{-1}(A_4)$  are in the 4-th quadrant.

Furthermore these 8 sets are disjoint since

$$S(A_1) \cap T(A_1) \neq \emptyset \Rightarrow \exists (x, y), (a, b) \in A_1 \text{ s.t. } \begin{cases} x + y = a \\ y = a + b \end{cases} \Rightarrow x = -b \Rightarrow (x, y), (a, b) \text{ cannot be both in } A_1.$$

Similar arguments apply to the other  $i$ 's. As a result we have

$$|E(A \setminus A_0, \bar{A})| \geq |A \setminus A_0| = |A| - |A_0|.$$

For  $(x, y) \in A_0$ , 2 of edges are self loops while the other 2 leave the axes. As a result

$$|E(A_0, \bar{A})| = |E(A_0, \bar{A}_0)| - |E(A_0, A \setminus A_0)| \geq 2|A_0| - 2|A \setminus A_0| = 4|A_0| - 2|A|.$$

Combine the two lower bounds to get

$$|E(A, \bar{A})| \geq \frac{2}{5}|A| \Rightarrow \phi(Z) \geq \frac{1}{10}.$$