Lecture 7: An Iterative Construction of Bipartite Expanders

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7.1 2-lift of a graph

Definition 7.1 (2-Lift of a Graph)**.** *Let $G = (V, E)$ be a graph. A* 2-lift *of G is a graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ defined as follows:*

- *The vertex set is a double copy of V:*

$$\widetilde{V} = V \times \{+1, -1\}.$$

- *For each edge $\{u, v\} \in E$, we choose a* sign *$\sigma(u, v) \in \{+1, -1\}$.*
  - *If $\sigma(u, v) = +1$, then we add parallel edges $(u, +1) \sim (v, +1)$, $(u, -1) \sim (v, -1)$ to $\widetilde{E}$.*
  - *If $\sigma(u, v) = -1$, then we add cross edges $(u, +1) \sim (v, -1)$, $(u, -1) \sim (v, +1)$ to $\widetilde{E}$.*

Thus the choice of signs $\{\sigma(u, v)\}_{(u,v)\in E}$ uniquely determines the 2-lift $\widetilde{G}$. Furthermore, $\widetilde{G}$ has twice the number of vertices of G and the same average degree as G.

The spectrum of G and that of $\widetilde{G}$ are closed related as evident from the following lemma.

Lemma 7.2. *Let $G = (V, E)$ be a graph with adjacency matrix $A \in \{0, 1\}^{n\times n}$. Fix a signing $\sigma : E \to \{\pm 1\}$ and let A^σ denote the* signed adjacency matrix, *i.e.*

$$A^\sigma_{uv} = \begin{cases} \sigma(u, v) & \text{if } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{G}$ be the 2-lift of G corresponding to the signing σ, with adjacency matrix $\widetilde{A}$. Then the multiset of eigenvalues of $\widetilde{A}$ is the disjoint union of the eigenvalues of A and the eigenvalues of A^σ.

Proof. Order the vertices of the 2-lift as $\{(v, +1) : v \in V\}$ followed by $\{(v, -1) : v \in V\}$. With this ordering, the adjacency matrix of the lift can we written as

$$\widetilde{A} = \frac{1}{2} \cdot \begin{bmatrix} A + A^\sigma & A - A^\sigma \\ A - A^\sigma & A + A^\sigma \end{bmatrix}.$$

Now observe that $\mathbb{R}^{2n}$ is the direct sum of the *symmetric* and *antisymmetric* subspaces:

$$U_+ = \{(x, x) : x \in \mathbb{R}^n\}, \qquad U_- = \{(x, -x) : x \in \mathbb{R}^n\}.$$

Case 1 (symmetric subspace). Take a vector $(x, x) \in U_+$. Then

$$\widetilde{A} \begin{bmatrix} x \\ x \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} A + A^\sigma & A - A^\sigma \\ A - A^\sigma & A + A^\sigma \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} Ax \\ Ax \end{bmatrix}.$$

Thus U_+ is invariant under $\widetilde{A}$ and the restriction of $\widetilde{A}$ to U_+ is exactly A.

Case 2 (antisymmetric subspace). Take a vector $(x, -x) \in U_-$. Then

$$\widetilde{A}\begin{bmatrix} x \\ -x \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} A + A^\sigma & A - A^\sigma \\ A - A^\sigma & A + A^\sigma \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} A^\sigma x \\ -A^\sigma x \end{bmatrix}.$$

Thus U_- is invariant under $\widetilde{A}$ and the restriction of $\widetilde{A}$ to U_- is exactly A^σ.

Since A is the adjacency matrix of G and A^σ is the signed adjacency matrix, the eigenvalues of $\widetilde{A}$ are precisely the union of the eigenvalues of A and of A^σ. □

7.2 An iterative construction of expanders

Bilu and Linial suggested the following way of constructing expanders via 2-lifts.

1. Start with a constant size d-regular expander G.
2. Randomly sign its edges, producing a signed adjacency matrix A^σ.
3. Iterating this process gives an infinite family of bounded-degree graphs.

They proved the following fact on the spectrum of a random 2-lift.

Theorem 7.3. *Let G be a d-regular graph with adjacency matrix A. There exists a signing $\sigma : E(G) \to \{\pm 1\}$ such that the signed adjacency matrix A^σ satisfies*

$$\|A^\sigma\|_{op} \;=\; O\!\left(\sqrt{d \log^3 d}\right).$$

So if G is a d-regular expander, its 2-lift $\widetilde{G}$ corresponding to σ is also a d-regular expander.

This existential result can be made algorithmic, and as a result Bilu and Linial gave an efficient algorithm that constructs an infinite family of d-regular expanders.

7.3 Existence of bipartite Ramanujan graphs

Inspired by this construction, Marcus–Spielman–Srivastava proved existence of an infinite family of **bipartite d-regular Ramanujan graphs**.

One can start with a constant size bipartite d-regular Ramanujan graph. Then it boils down to prove that there exists a signing σ such that the eigenvalues of A^σ are in $[-d, 2\sqrt{d-1}]$.

Theorem 7.4. *For every d-regular graph G, there exists a signing σ such that the eigenvalues of the signed adjacency matrix A^σ are all contained in the interval*

$$[-d,\, 2\sqrt{d-1}].$$

Hence, if G is a bipartite d-regular Ramanujan graph, then the corresponding 2-lift is also a bipartite d-regular Ramanujan graph.

To prove the theorem, first recall that the eigenvalues of A^σ are the roots of the characteristic polynomial $\det(xI - A^\sigma)$. A classical result due to Godsil stated that the expectation of the characteristic polynomial over random signing is the matching polynomial of G.

Lemma 7.5 (Godsil)**.** *Let $G = (V, E)$ be a graph on n vertices. Then*

$$\mathbb{E}_\sigma[\det(xI - A^\sigma)] = \mu_G(x),$$

where

$$\mu_G(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k \, x^{n-2k},$$

and m_k is the number of k-matchings in G.

Proof. Expand the determinant as:

$$\det(xI - A^\sigma) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} (xI - A^\sigma)_{i,\pi(i)}.$$

Now take the expectation over random signing. Any monomial in the expansion is a product of entries of A^σ and a power of x. Because the signs are independent and have mean zero, the expectation vanishes unless every edge appears an even number of times.

The only permutations π that survive this averaging are those consisting of disjoint transpositions and fixed points:

$$\pi = (i_1 \, j_1)(i_2 \, j_2) \cdots (i_k \, j_k),$$

corresponding to a k-matching in G. In such a π, each fixed point contributes a factor of x, while each transposition (i, j) contributes a factor of -1 (to $\operatorname{sgn}(\pi)$).

Thus, a matching of size k contributes $(-1)^k x^{n-2k}$, and the number of such terms is exactly the number of k-matchings in G.

So, summing over all k eventually gives

$$\mathbb{E}_\sigma[\det(xI - A^\sigma)] = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k \, x^{n-2k} = \mu_G(x).$$

□

The Heilmann–Lieb theorem on matching polynomials says that for every d-regular graph G, the roots of $\mu_G(x)$ are in $[-2\sqrt{d-1}, \, 2\sqrt{d-1}]$.

Marcus–Spielman–Srivasta then used the method of interlacing polynomials to prove that there exists a signing σ such that the characteristic polynomial $\det(xI - A^\sigma)$ has its maximum root bounded by that of the average polynomial $\mu_G(x)$. To explain it we start with defining interlacing polynomials.

Definition 7.6 (Interlacing polynomial)**.** *We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ interlaces a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if*

$$\beta_n \le \alpha_{n-1} \le \beta_{n-1} \le \alpha_{n-2} \le \cdots \le \alpha_1 \le \beta_1.$$

Furthermore, we say that polynomials $f_1, \ldots, f_k$ have a common interlacing if there is a polynomial g such that g interlaces f_i for every $i \in [k]$.

Lemma 7.7. *Let $f_1, \ldots, f_k$ be polynomials of the same degree that are real-rooted and have positive leading coefficients. Define*

$$f = \sum_{i=1}^{k} f_i$$

If $f_1, \ldots, f_k$ have a common interlacing, then there exists an $i \in [k]$ so that the largest root of f_i is at most the largest root of f.

Proof. Let n be the degree of f_i and g be the common interlacing with roots $\alpha_{n-1} \leq \cdots \leq \alpha_1$. Since each f_i has a positive leading coefficient, $f_i(x) \geq 0$ for $x \geq \beta_1^{(i)}$ which is the top root of f_i. Furthermore, each f_i has only one root that is $\geq \alpha_1$, so $f_i(\alpha_1) \leq 0$. Subsequently $f(\alpha_1) \leq 0$. This tells us that f has a root that is at least α_1. Let β_1 be the largest root of f. Since f is the sum of the f_i, there must be some i for which $f_i(\beta_1) \geq 0$. As f_i only has one root $\beta_1^{(i)} \geq \alpha_1$, then $\alpha_1 \leq \beta_1^{(i)} \leq \beta_1$. □

So now it suffices to prove that the characteristic polynomials have a common interlacing. We do so recursively, by first noting that

$$f_+(x) + f_-(x) = 2^{|E|} \cdot \mu_G(x)$$

where

$$f_+(x) = \sum_{\sigma:\sigma(1)=+} p_\sigma(x), \qquad f_-(x) = \sum_{\sigma:\sigma(1)=-} p_\sigma(x).$$

Then we are going to show that f_+ and f_- have a common interlacing. Thus one of the function has maximum root bounded by $2\sqrt{d-1}$. Then recursively decompose the function with bounded maximum root until ending with a p_σ with bounded maximum root.

To prove that the common interlacing exists, we need the following two results. Their proofs will be skipped.

Lemma 7.8. *Let f, g be univariate polynomials of the same degree with positive leading coefficients. Then f, g have a common interlacing if and only if for all convex combinations $h_t(x) = tf(x) + (1-t)g(x)$ where $t \in [0,1]$ is real-rooted.*

Lemma 7.9. *Consider a graph G with m edges. For any $c_1, \ldots, c_m \in [0,1]^m$, the following polynomial is real-rooted*

$$\sum_{\sigma \in \{\pm\}^m} \prod_{i:\sigma(i)=+} c_i \cdot \prod_{i:\sigma(i)=-} (1-c_i) \cdot p_\sigma(x).$$

Finally as a corollary we get that:

Corollary 7.10 (Restatement of Theorem 7.4)**.** *Let $G = (V, E)$ be a d-regular graph. For each signing $\sigma : E \to \{\pm 1\}$, let $p_\sigma(x)$ be the characteristic polynomial of the signed adjacency matrix A^σ. Then there exists a signing σ such that the largest root of $p_\sigma(x)$ is at most $2\sqrt{d-1}$.*