590.05 Expanders and High-dimensional Expanders **Fall 2025**

Lecture 8: Expanders from Zig-zag Products

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8.1 Zig–zag product of graphs

Definition 8.1 (Zig–Zag Product)**.** *Let G be a D-regular graph on vertex set $[N]$, with its edges labeled by $[D]$. Let H be a d-regular graph on vertex set $[D]$.*

The zig–zag product $G \textcircled{z} H$ *is the d^2-regular graph on vertex set*

$$V(G \textcircled{z} H) = [N] \times [D].$$

An edge is defined as follows. From a vertex $(v, i) \in [N] \times [D]$:

1. ***Zig:*** *Move inside H from i to j, i.e. $(v, i) \to (v, j)$.*
2. ***Zag:*** *Cross the edge of G incident to v labeled j, arriving at (u, j).*
3. ***Zig:*** *Move inside H again, from j to k, arriving at (u, k).*

Thus each edge of $G \textcircled{z} H$ is of the form

$$(v, i) \longrightarrow (u, k),$$

where (i, j) and (j, k) are edges of H, and (v, u) is the adjacent edge of v labeled j.

8.2 Zig–zag product expanders

In this section we prove that zig–aag products preserve expansion of the two original graphs.

Theorem 8.2 (Reingold–Vadhan–Wigderson)**.** *Let G be a D-regular graph on N vertices with second-largest eigenvalue λ_G, and let H be a d-regular graph on D vertices with second-largest eigenvalue in absolute value λ_H. Then the zig–zag product $G \textcircled{z} H$ is a d^2-regular graph on $N \cdot D$ vertices whose second-largest eigenvalue satisfies*

$$\lambda(G \textcircled{z} H) \le \lambda_G + \lambda_H + \lambda_H^2.$$

Proof. Let A_G be the normalized adjacency matrix of G (an $N \times N$ matrix), and A_H the normalized adjacency matrix of H (a $D \times D$ matrix).

Define the permutation matrix P of size $ND \times ND$ that performs the "zag" step: it maps (v, j) to (u, j) where (v, u) is the edge of G labeled j. The "zig" steps are implemented by $I_N \otimes A_H$, acting on $[N] \times [D]$.

Thus the normalized adjacency matrix of $G \textcircled{z} H$ is

$$A = (I_N \otimes A_H)\, P\, (I_N \otimes A_H).$$

Now do a spectral decomposition of A_H as

$$A_H = \tfrac{1}{D}J + B,$$

where J is the all-ones matrix and $\|B\| \le \lambda_H$. Substitute the decomposition:

$$A = (I \otimes (\tfrac{1}{D}J + B))\, P\, (I \otimes (\tfrac{1}{D}J + B)).$$

Expanding gives four terms:

$$A = (I \otimes \tfrac{1}{D}J)\, P\, (I \otimes \tfrac{1}{D}J) + (I \otimes B)\, P\, (I \otimes \tfrac{1}{D}J) + (I \otimes \tfrac{1}{D}J)\, P\, (I \otimes B) + (I \otimes B)\, P\, (I \otimes B).$$

For every $x \perp \vec{1}$ in $\mathbb{R}^{ND}$ we decompose $x = x_\parallel \otimes \vec{1}_D + x_\perp$ where

$$x_\parallel(v) = \frac{1}{D}\sum_{j=1}^{D} x(v,j)\,, \quad x_\perp(v,i) = x(v,i) - x_\parallel(v).$$

Notice that for all v

$$\sum_{i=1}^{D} x_\perp(v,i) = \sum_{i=1}^{D} x(v,i) - \sum_{i=1}^{D} x_\parallel(v,i) = 0.$$

Therefore $x_\perp(v) \perp \vec{1}_D$ and $x_\parallel \perp \vec{1}_N$. As a result

$$(I \otimes \tfrac{1}{D}J)\, x_\perp = \vec{0}, \quad (I \otimes B)\, x_\parallel \otimes \vec{1}_D = \vec{0}$$

$$(I \otimes \tfrac{1}{D}J)\, x_\parallel \otimes \vec{1}_D = x_\parallel \otimes \vec{1}_D, \quad (I \otimes B)\, x_\perp = \begin{bmatrix} Bx_\perp(v_1) \\ \vdots \\ Bx_\perp(v_N) \end{bmatrix}$$

Then for each of the four terms we can get that

$$\left\langle x, (I \otimes \tfrac{1}{D}J)\, P\, (I \otimes \tfrac{1}{D}J) x \right\rangle = \left\langle x_\parallel \otimes \vec{1}_D, P x_\parallel \otimes \vec{1}_D \right\rangle = D \cdot \left\langle x_\parallel, A_G x_\parallel \right\rangle \le \lambda_G \|x_\parallel \otimes \vec{1}_D\|_2^2$$

$$\begin{aligned} \left\langle x, (I \otimes B)\, P\, (I \otimes \tfrac{1}{D}J) x \right\rangle &= \left\langle x_\perp, (I \otimes B)\, P\, (I \otimes \tfrac{1}{D}J)\, x_\parallel \otimes \vec{1}_D \right\rangle \\ &= \begin{bmatrix} Bx_\perp(v_1) & \dots & Bx_\perp(v_N) \end{bmatrix}\, P\, x_\parallel \otimes \vec{1}_D \\ &\le \lambda_H \|x_\perp\|_2 \cdot \left\| x_\parallel \otimes \vec{1}_D \right\|_2 \end{aligned}$$

$$\langle x, (I \otimes B)\, P\, (I \otimes B) x \rangle = \begin{bmatrix} Bx_\perp(v_1) & \dots & Bx_\perp(v_N) \end{bmatrix}\, P \begin{bmatrix} Bx_\perp(v_1) \\ \vdots \\ Bx_\perp(v_N) \end{bmatrix} \le \lambda_H^2 \|x_\perp\|_2^2 .$$

Thus, combining all four terms together, we have that for every x in the space orthogonal to $\vec{1}$, the operator norm of A is bounded by

$$\frac{\langle x, Ax \rangle}{\|x\|_2^2} \le \frac{\lambda_G \left\| x_\parallel \otimes \vec{1}_D \right\|_2^2 + 2\lambda_H \left\| x_\parallel \otimes \vec{1}_D \right\|_2 \cdot \|x_\perp\|_2 + \lambda_H^2 \|x_\perp\|_2^2}{\left\| x_\parallel \otimes \vec{1}_D \right\|_2^2 + \|x_\perp\|_2^2} \le \lambda_G + \lambda_H + \lambda_H^2.$$

□

As a result we can construct infinite families of Δ-regular graphs in the following way.

1. Start with a $\sqrt{\Delta}$-regular graph H over Δ^2 vertices.
2. Let $G_1 = H^2$ then G_1 has degree Δ and expansion λ_H^2.
3. Iteratively construct $G_{i+1} = G_i^2 \text{ z } H$. Note that this is a valid zig–zag product since the degree of G_i^2 and the vertex number in H are both Δ^2.

Note that if we want to construct λ-expanders, then it suffices to pick H to be λ-expander. Since in this case

$$\lambda(G_1) = \lambda_H^2 \leq \lambda^2, \quad \lambda(G_{i+1}) \leq \lambda(G_i)^2 + \lambda_H(1+\lambda_H) \leq \lambda^2 + \lambda(1+\lambda) \leq \lambda.$$

8.3 Expansion of $G(n,p)$

As a final part of expanders, we use trace method to compute the expansion of Erdös-Renyí graphs.

Theorem 8.3. *Let $G \sim G(n,p)$ be an Erdős–Rényi random graph on n vertices. Assume $np \geq 2\log^8 n$. Then with high probability*

$$|\lambda|_2(G) \;\leq\; (1+o_n(1)) \cdot 2\sqrt{np},$$

where $|\lambda|_2(G)$ denotes the second-largest eigenvalue in absolute value of the adjacency matrix of G.

Proof. The adjacency matrix A of $G(n,p)$ can be decomposed as

$$A \;=\; \mathbb{E}[A] + R,$$

where $\mathbb{E}[A] = p(J-I)$ and $R = A - \mathbb{E}[A]$ is a centered random symmetric matrix with independent entries. We further observe the spectral connection between A and R:

$$|\lambda|_2(G) \leq \|A - pJ\|_{op} \leq \|A - \mathbb{E}[A]\|_{op} = \|R\|_{op}.$$

Thus it suffices to show w.h.p.

$$\|R\|_{op} = (1+o(1))2\sqrt{np}.$$

Trace moment method. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of R. For any even integer $2k$,

$$\|R\|_{op}^{2k} \;\leq\; \mathrm{Tr}(R^{2k}) = \sum_{i=1}^{n} \lambda_i^{2k}.$$

Taking expectation,

$$\mathbb{E}\,\mathrm{Tr}(R^{2k}) = \sum_{i=1}^{n} \mathbb{E}[\lambda_i^{2k}].$$

Combinatorial interpretation. We can expand trace moments as sum of weighted walks

$$\mathrm{Tr}(R^{2k}) = \sum_{v_0, v_1, \ldots, v_{2k-1}} R_{v_0 v_1} R_{v_1 v_2} \cdots R_{v_{2k-1} v_0}.$$

Taking expectation, independence of entries implies that only terms corresponding to closed walks in which every edge is traversed at least twice (no singleton edges) survive. Each such walk uses at most $k+1$ vertices and contributes at most p^k to the expectation.

Bounding the number of closed walks. The number of closed walks of length $2k$ without a singleton edge on exactly s vertices is bounded by

$$C_s \le n^s \binom{2k}{s-1}\binom{2k-(s-1)}{s-1} s^{2k-2(s-1)}$$

- n^s: choose the s vertices $v_1, \ldots, v_s$ in the walk.
- $\binom{2k}{s-1}$: location of steps reaching a new vertex. Call these edges the tree edges.
- $\binom{2k-(s-1)}{s-1}$: location of second time traversing the tree edges.
- $s^{2k-2(s-1)}$: all other steps and the visited vertices they are reaching.

In the case $s = k+1$

$$n^s \binom{2k}{s-1}\binom{k}{s-1} \cdot s^{2k-2(s-1)} \le n^{k+1} 2^{2k}$$

For all $s = 2 \ldots, k$

$$\frac{C_s}{C_{s+1}p} \le \frac{1}{np} \cdot \frac{s}{(2k-s+1)} \cdot \frac{s(2k-s+1)}{(2k-2s+2)(2k-2s+1)} \cdot (s+1)^2 \le \frac{k^2(s+1)^2}{np} \le O\left(\frac{1}{\log^4 n}\right).$$

The last inequality follows from setting $k = \Theta(\log n)$ and $p \ge \frac{\log^8 n}{n}$. Therefore in the trace moment calculation the term with $s = k+1$ dominates,

$$\mathbb{E}\,\mathrm{Tr}(R^{2k}) \le \sum_{s=2}^{k+1} C_s p^{s-1} = \left(1 + O\left(\frac{1}{\log^4 n}\right)\right) C_{k+1} p^k \le (1+o(1)) \cdot n(4np)^k.$$

Markov's inequality. Set $k = c \log n$ for a small constant $c > 0$. Then by Markov's inequality,

$$\Pr[\|R\| \ge (1+\varepsilon) 2\sqrt{np}] \;\le\; \frac{\mathbb{E}\,\mathrm{Tr}(R^{2k})}{((1+\varepsilon)^{2k}(4np)^k)}.$$

Plugging in the bound gives

$$\Pr[\|R\| \ge (1 + 1/o(\log n)) 2\sqrt{np}] \le e^{-\omega(1)}.$$

Therefore, with high probability

$$\|R\| \le (1+o(1)) \cdot 2\sqrt{np}.$$

This gives

$$\lambda_2(G) = (1+o(1)) \cdot 2\sqrt{np}.$$

□