590.05 Expanders and High-dimensional Expanders	**Fall 2025**

Lecture 9: Introduction to High-dimensional Expanders

Instructor: Siqi Liu *Scribe: Siqi Liu*

9.1 Simplicial complexes

We generalize expansion to hypergraphs $(V, \mathcal{E})$, where $\mathcal{E}$ is a set of hyperedges.

Definition 9.1 (Simplicial complex)**.** *A simplicial complex X is a hypergraph with downward closure. I.e. for every hyperedge $\tau \in X$, the subedges $\sigma \subseteq s$ are also in X.*

We denote the set hyperedges of size $(i+1)$ as $X(i) \subseteq \binom{V}{i+1}$. A hyperedge $\sigma \in X(i)$ is also called an i-face.

The dimension d of X is the largest i such that $X(i) \neq \varnothing$.

We will mainly consider pure simplicial complexes.

Definition 9.2 (Pure simplicial complex)**.** *A d-dimensional simplicial complex X is* pure *if for every $\sigma \in \bigcup_{i=0}^{d-1} X(i)$ there exists a $\tau \in X(d)$ such that $\sigma \subseteq \tau$. In other words, maximal faces induce all lower-dimensional faces.*

Example 9.3. *Here is an example of a pure 2-dimensional simplicial complex presented in two ways.*

$$X(0) = \{a, b, c, d, e, f\}, \quad X(1) = \{ab, ad, bd, cd, cf, df, be, de\}, \quad X(2) = \{abd,\ cdf,\ bde\}.$$

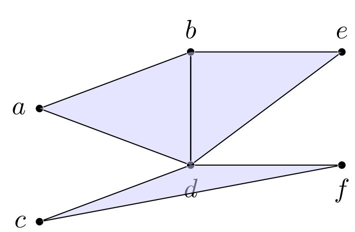

Figure 9.1: X

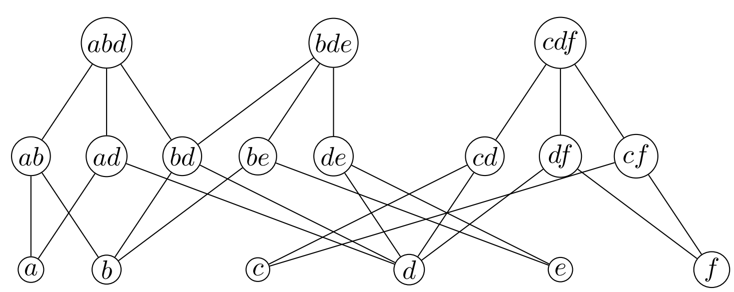

Figure 9.2: Hasse diagram

9.2 Links, skeletons, and their random walks

Definition 9.4 (Link)**.** *In a d-dimensional simplicial complex X, for every $i \leq d-2$ and $\sigma \in X(i)$, the* link *of σ is the $(d-i-1)$-dimensional simplicial complex*

$$X_\sigma = \{\, \tau \setminus \sigma \mid \tau \in X,\ \tau \supseteq \sigma \,\}.$$

Example 9.5. *Consider X from Example 9.3:*

$$X_a = \{b, d, \{b, d\}\}, \quad X_d = \{a, b, c, e, f, \{a, b\}, \{b, e\}, \{c, f\}\}.$$

Definition 9.6 (Skeleton)**.** *For a d-dimensional simplicial complex X and some $k \leq d$, the k-skeleton of X is the simplicial complex*

$$\bigcup_{i=-1}^{k} X(i),$$

consisting of all faces of dimension at most k.

Random walks on simplicial complexes

To define random walks consistently, we assign weights to all faces of a pure simplicial complex X.

Definition 9.7 (Weight function)**.** *Let X be a pure d-dimensional simplicial complex with a weight function $w_d : X(d) \to \mathbb{R}_{\geq 0}$. For every $i = 0, \ldots, d-1$, iteratively define the induced weight on i-faces as*

$$w_i(\sigma) \;=\; \sum_{\tau \in X(i+1): \sigma \subseteq \tau} w_{i+1}(\tau), \quad \forall \sigma \in X(i).$$

Thus for i-face σ, its weight is the sum of the weights of $(i+1)$-faces containing it.

The link of $\sigma \in X(i)$ inherits the weight from X as follows

$$\forall \tau \in X_\sigma(j), w_{\sigma,j} = w_{i+j+1}(\sigma \cup \tau).$$

Given the weight functions, we can define the random walk on the 1-skeleton of X as

$$\Pr[v_1 = u \mid v_0 = v] = \frac{w_1(\{u, v\})}{w_0(v)}.$$

Analogously the walk the random walk on the 1-skeleton of X_σ is

$$\Pr[v_1 = u \mid v_0 = v] = \frac{w_{\sigma,1}(\{u, v\})}{w_{\sigma,0}(v)} = \frac{w_{i+2}(\sigma \cup \{u, v\})}{w_{i+1}(\sigma \cup \{v\})}.$$

9.3 High-dimensional expanders

We want to generalize spectral expansion to simplicial complexes. In a simplicial complex, we can define expansion of 1-skeletons now that we have defined random walks on them.

Definition 9.8 (High-dimensional expanders)**.** *A pure d-dimensional simplicial complex X (with weight function w_d) is a one-sided (or two-sided) γ-expander if:*

1. *The 1-skeleton is a one-sided (or two-sided) γ-expander.*
2. *For all $i \leq d-1$ and $\sigma \in X(i)$, the 1-skeleton of X_σ is a one-sided (or two-sided) γ-expander.*

Remark 9.9. *A quick check give that d-dim complete complexes are d-dim two-sided $\frac{1}{n-d-2}$-expanders while complete d-dim partite complexes are d-dim one-sided 0-expanders.*