

1 Introduction

Expander graphs are families of sparse graphs that nevertheless exhibit strong connectivity properties. In the classical, regular setting, one works with a fixed integer $d \geq 3$ and considers a family of finite d -regular graphs $\{G_n\}$ with $|V(G_n)| \rightarrow \infty$. The Alon-Boppana bound shows that there is a universal obstruction to making $\lambda_2(G_n)$, the second largest eigenvalue of the adjacency matrix of G_n , too small:

$$\lambda_2(G_n) \geq 2\sqrt{d-1} - o_n(1). \quad (1)$$

Thus, for fixed d and any $\epsilon > 0$, no infinite family of d -regular graphs can satisfy $\lambda_2(G_n) \leq 2\sqrt{d-1} - \epsilon$. Graphs whose nontrivial eigenvalues all lie in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ are called *Ramanujan graphs* and are regarded as optimally expanding.

The constant $2\sqrt{d-1}$ is not random; it is the spectral radius of the infinite d -regular tree T_d , the *universal covering* (defined later) graph of every finite d -regular graph. Let A_{T_d} be the adjacency operator on the Hilbert space $\ell^2(V(T_d))$. Its *spectral radius*, akin to that of a finite matrix, is defined by the Rayleigh quotient

$$\rho(T_d) = \sup_{0 \neq f \in \ell^2(V(T_d))} \frac{\langle f, A_{T_d} f \rangle}{\langle f, f \rangle}. \quad (2)$$

Kesten [\[link\]](#) showed that $\rho(T_d) = 2\sqrt{d-1}$. In this notation, the Alon-Boppana bound can be re-written as

$$\lambda_2(G_n) \geq \rho(T_d) - o_n(1), \quad (3)$$

so the nontrivial spectrum of any large d -regular graph must accumulate at the spectral radius of its universal cover.

The goal of this write-up is to understand the corresponding picture for *irregular* graphs. When the degree is not constant, there is no single parameter d and the expression $2\sqrt{d-1}$ has no direct meaning. However, it can be proven that every finite connected graph G still has a universal cover T , and consequently the notion $\rho(T)$ is still well-defined. The **Greenberg-Lubotzky** theorem is a generalization of the Alon-Boppana bound to irregular graphs. It asserts that for any family of finite graphs whose universal covers all coincide with the same tree T , the nontrivial eigenvalues of their adjacency matrices cannot avoid $\rho(T)$. In other words, $\rho(T)$ plays exactly the role $\rho(T_d)$ plays in the regular case; it is the universal lower bound for how small the nontrivial spectrum can be in a large family of quotients of T .

2 Preliminaries: Universal Covers & Spectral Radius

We first collect some basic objects that will appear in the Greenberg-Lubotzky theorem: graph coverings, the universal cover of a finite graph, and the spectral radius of the adjacency operator on this infinite cover. Throughout, we adopt notations used throughout the CS590 lecture notes: all graphs in this note are finite, connected, undirected, and without loops or multiple edges, unless stated otherwise. For $G = (V, E)$ we write $|V| = n$. We usually assume that the degrees are uniformly bounded, so that all constructions below are well behaved.

2.1 Graphs & Adjacency Operators

The (combinatorial) adjacency operator of G is the linear map

$$A_G : \mathbb{R}^V \rightarrow \mathbb{R}^V : \quad (A_G f)(v) = \sum_{(u,v) \in E} f(u), \quad (4)$$

where the sum runs over neighbors u of v . When $V = \{1, \dots, n\}$, this is the usual $n \times n$ adjacency matrix. Since A_G is real symmetric, it admits an orthonormal basis of eigenvectors and has real eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$. For a connected d -regular graph one has $\lambda_1(G) = d$; in that setting, the *nontrivial spectrum* refers to $\lambda_2(G), \dots, \lambda_n(G)$, and the Alon-Boppana bound shows that, for any infinite family of such graphs, the nontrivial eigenvalues must accumulate at $2\sqrt{d-1}$.

In the irregular case, the top eigenvalue and its eigenvector no longer have this simple description, and it is more natural to compare the spectrum of G with the spectrum of an infinite graph associated to it: its *universal cover*.

2.2 Coverings & the Universal Cover

Definition 2.1 (Covering Map). Let $G = (V, E)$ and $\tilde{G} = (\tilde{V}, \tilde{E})$ be connected graphs. A map $\pi : \tilde{V} \rightarrow V$ is called a **covering map** if (i) it is surjective, and (ii) for every vertex $\tilde{v} \in \tilde{V}$, the restriction of π induces a bijection between the neighbors of \tilde{v} in \tilde{G} and the neighbors of $\pi(\tilde{v})$ in G .

In this situation, \tilde{G} is called a **cover** of G , and G is called a **quotient** of \tilde{G} .

The simplest example is the cyclic cover: the cycle C_{2m} covers C_m via the map that sends vertices i and $i + m$ to of the larger $2m$ -cycle to vertex i of the smaller m -cycle.

The collection of all covers of a fixed connected graph G forms a partially ordered set under the relation “covers;” among them, there is a maximal object, the *universal cover*, which we formalize now.

Definition 2.2 (Universal Cover). Let G be a connected finite graph. A **universal cover** of G is a tree T equipped with a covering map $T \rightarrow G$ such that for any other $\pi' : H \rightarrow G$, there exists a unique covering map $\varphi : T \rightarrow H$ such that $\pi = \pi' \circ \varphi$.

It is well known that the universal cover exists and is unique up to graph isomorphism. In particular, any connected finite graph admits a canonical infinite tree T that captures all of its finite covering graphs simultaneously. For the purposes of the Greenberg-Lubotzky theorem, it is convenient to have a concrete model of T in terms of walks.

Definition 2.3 (Non-backtracking Walks & the Universal Cover). Fix a base vertex $v_0 \in V(G)$. A walk $w = (v_0, v_1, \dots, v_k)$ is called **non-backtracking** if $v_{i+2} \neq v_i$ whenever the two vertices are defined. That is, the walk never *immediately* traverses an edge then returns across it.

For a fixed $v_0 \in V(G)$, the universal cover T of G can be realized as follows.

- (Vertices) The vertex set $V(T)$ consists of all non-backtracking walks in G starting at v_0 .
- (Edges) Two walks are adjacent in T if one walk is a single-step extension of the other.
- (Covering map) The covering map $\pi : T \rightarrow G$ sends a walk to its terminal vertex.

In this model, the tree structure of T is transparent: extending a walk by one step cannot create a cycle in T , and every vertex in T has exactly one “parent” and finitely many “children” determined by its endpoint in the original graph G . If G is d -regular, every vertex in T has degree d ; the universal cover in this case is then the infinite d -regular tree, denoted T_d .

Finally, for convenience we fix notation for local neighborhoods by drawing connections to real analysis. If X is a graph and $x \in V(X)$, the **ball** of radius r around x , denoted $B_X(x, r)$ or $B(x, r)$, is the induced subgraph spanned by all vertices at graph distance at most r from x .

The main feature of a covering map $\pi : T \rightarrow G$ is that *every ball* $B_T(\tilde{v}, r)$ *maps isomorphically onto the ball* $B_G(\pi(\tilde{v}), r)$. In particular, all finite-radius neighborhoods in T look like “unwrapped” versions of neighborhoods in G .

2.3 Spectral Radius of the Universal Cover

The universal cover T is generally infinite, so its adjacency operator acts on an infinite-dimensional space. Let $V(T)$ be the vertex set of the universal cover. Consider the Hilbert space $\ell^2(V(T))$ of square-summable real functions on $V(T)$,

$$\ell^2(V(T)) = \left\{ f : V(T) \rightarrow \mathbb{R} : \sum_{x \in V(T)} f(x)^2 < \infty \right\} \quad (5)$$

equipped with the usual inner product $\langle f, g \rangle = \sum_x f(x)g(x)$. The adjacency operator

$$A_T = \ell^2(V(T)) \rightarrow \ell^2(V(T)), \quad (A_T f)(x) = \sum_{y \sim x} f(y) \quad (6)$$

is a bounded self-adjoint operator whenever T has uniformly bounded degrees. Its operator norm is the natural notion of “spectral radius” for the universal cover. (The notation $y \sim x$ just means y is adjacency to x in the universal cover T .)

Definition 2.4 (Spectral Radius of the Universal Cover). The spectral radius of T is defined as

$$\rho(T) = \|A_T\|_{\text{op}} = \sup_{0 \neq f \in \ell^2(V(T))} \frac{\langle f, A_T f \rangle}{\langle f, f \rangle}. \quad (7)$$

In the regular case, this reproduces the constant appearing in the Allon-Boppana bound: Kesten [link] showed that the spectrum of A_{T_d} is the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$; in particular $\rho(T_d) = 2\sqrt{d-1}$.

For an arbitrary finite graph G with universal cover T , the Greenberg-Lubotzky theorem will assert that $\rho(T)$ plays the same role: it is a universal asymptotic lower bound. We present this result following Hoory-Linial-Wigderson in the next section.

3 The Greenberg-Lubotzky Theorem

Let G be a finite connected graph with adjacency matrix A_G . Write the eigenvalues of A_G as $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$, and recall that $\lambda_1(G)$ is simple and has a positive eigenvector by Perron-Frobenius.

Definition 3.1 (Nontrivial Spectral Radius). The **nontrivial spectral radius** of G is $\lambda(G) = \max_{2 \leq i \leq n} |\lambda_i(G)|$. This is the quantity that governs expansion.

Theorem 3.2 (Greenberg-Lubotzky). Let $\{G_i\}_{i=1}^{\infty}$ be a family of finite connected graphs with $|V(G_i)| \rightarrow \infty$, all covered by the same universal cover T . Then,

$$\liminf_{i \rightarrow \infty} \lambda(G_i) \geq \rho(T). \quad (8)$$

Equivalently, for every $\epsilon > 0$, all but finitely many graphs G_i satisfy $\lambda(G_i) \geq \rho(T) - \epsilon$. This generalizes the Allon-Boppana lower bound from the d -regular case, where $T = T_d$ and $\rho(T_d) = 2\sqrt{d-1}$.

The proof of Theorem 3.2 is rather long and divided into multiple parts. For consistency, we follow the outline and notations in **HLW** and indicate overlap with lecture materials when applicable.

3.1 Rayleigh Quotients & Closed Walks in Finite Graphs

Let G be a finite connected graph with adjacency matrix A . For each integer $\ell \geq 0$, the (u, v) entry of A^ℓ counts the number of length- ℓ walks from u to v . Consequently, $(A^{2k})_{u,u} = C_G(2k, u)$ counts the number of closed, length- $2k$ walks starting and ending at u . We will need the variational description of $\lambda(G)$ in terms of Rayleigh quotients and the “two-point” test vector used in the proof of Alon-Boppana.

Lemma 3.3 (Rayleigh Quotient for $\lambda(G)$). Let G, A be given with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then, for every integer $k \geq 1$,

$$\lambda(G)^{2k} = \max\{(f^\top A^{2k} f) / \|f\|^2 : f \perp \mathbf{1}\}. \quad (9)$$

Proof. Since A is real symmetric, we can choose an orthonormal eigenbasis $\{v_1, \dots, v_n\}$ with $Av_j = \lambda_j v_j$ and $v_1 = \mathbf{1}/\sqrt{n}$. Write f as $f = \sum_{j=1}^n a_j v_j$. Then $f \perp \mathbf{1}$ if and only if $a_1 = 0$. Moreover, $A^{2k} f = \sum_{j=1}^n a_j \lambda_j^{2k} v_j$, so

$$\frac{f^\top A^{2k} f}{\|f\|^2} = \frac{\sum_{j=1}^n |a_j|^2 \lambda_j^{2k}}{\sum_{j=1}^n |a_j|^2}, \quad (10)$$

If $a_1 = 0$, this becomes $(\sum_{j \geq 2} |a_j|^2 \lambda_j^{2k}) / (\sum_{j \geq 2} |a_j|^2)$, whose maximum over the choices of $(a_j)_{j \geq 2}$ is clearly $\max_{j \geq 2} |\lambda_j|^{2k} = \lambda(G)^{2k}$. \square

Lemma 3.4 (Two-point Test Vector). *Let G be a finite graph of diameter Δ , and let $s, t \in V(G)$ satisfy $d_G(s, t) = \Delta$. Fix an integer k with $2k < \Delta$, and define $f = \delta_s - \delta_t$, where δ_x is the indicator of vertex x . Then,*

$$\frac{f^\top A^{2k} f}{\|f\|^2} = \frac{(A^{2k})_{s,s} + (A^{2k})_{t,t} - (A^{2k})_{s,t}}{2} = \frac{C_G(2k, s) + C_G(2k, t)}{2}, \quad (11)$$

and hence

$$\lambda(G)^{2k} \geq \frac{C_G(2k, s) + C_G(2k, t)}{2}. \quad (12)$$

Proof. We have $\|f\|^2 = 1^2 + (-1)^2 = 2$, and $f^\top A^{2k} f = \sum_{u,v} f(u)(A^{2k})_{u,v}f(v) = (A^{2k})_{s,s} + (A^{2k})_{t,t} - 2(A^{2k})_{s,t}$. Then entry $(A^{2k})_{s,t}$ counts the number of length- $2k$ walks from s to t . If $2k < d_G(s, t) = \Delta$ then there is no such walk, so $(A^{2k})_{s,t} = 0$. Thus

$$\frac{f^\top A^{2k} f}{\|f\|^2} = \frac{(A^{2k})_{s,s} + (A^{2k})_{t,t}}{2} = \frac{C_G(2k, s) + C_G(2k, t)}{2}. \quad (13)$$

Since $f \perp \mathbf{1}$, Lemma 3.3 implies that $\lambda(G)^{2k} \geq f^\top A^{2k} f / \|f\|^2$, giving the inequality. \square

3.2 Coverings and Closed Walk Counts

Now let $\pi : T \rightarrow G$ be a covering map from an infinite tree T (e.g. the universal cover) onto a finite connected graph G . We will show that closed-walk counts do not increase when passing to a quotient in the following lemma.

Lemma 3.5 (Closed Walks Inject Under the Projection). *Let $\pi : T \rightarrow G$ be a covering map, $u \in V(G)$, and $\tilde{u} \in V(T)$ with $\pi(\tilde{u}) = u$. Then, for each length $\ell \geq 0$:*

- Every walk $w = (u = v_0, \dots, v_\ell)$ in G has a unique lift $\tilde{w} = (\tilde{u} = \tilde{v}_0, \dots, \tilde{v}_\ell)$ in T with $\pi(\tilde{v}_i) = v_i$ for all i ;
- The map Φ_ℓ mapping closed walks of length ℓ at \tilde{u} to closed walks of length ℓ at u , given by projecting each vertex via π , is injective.

In particular, for every even length $2k$, we have $C_G(2k, u) \geq C_T(2k, \tilde{u})$.

Proof. For (i), we prove by induction on ℓ that every walk in G starting at u lifts uniquely once we fix the starting lift \tilde{u} . For $\ell = 0$ there is nothing to prove. Suppose the claim holds for length ℓ and let $w = (u = v_0, \dots, v_\ell, v_{\ell+1})$ be a walk of length $\ell + 1$. By induction its prefix (v_0, \dots, v_ℓ) has a unique lift $(\tilde{v}_0 = \tilde{u}, \dots, \tilde{v}_\ell)$. Since $v_{\ell+1}$ is a neighbor of v_ℓ and π is a bijection, there is a unique neighbor $\tilde{v}_{\ell+1}$ of \tilde{v}_ℓ with $\pi(\tilde{v}_{\ell+1}) = v_{\ell+1}$. This extends the lift uniquely.

For (ii), let \tilde{w}_1, \tilde{w}_2 be two closed walks at \tilde{u} of length ℓ with the same projection w in G . Then both \tilde{w}_1, \tilde{w}_2 are lifts of the same walk starting at \tilde{u} . By uniqueness of lifts, $\tilde{w}_1 = \tilde{w}_2$. Thus Φ_ℓ is injective.

The injectivity of Φ_{2k} immediately implies $C_G(2k, u) \geq C_T(2k, \tilde{u})$. \square

3.3 Closed Walks and the Spectral Radius on Universal Cover

Let T be the universal cover, with adjacency operator $A_T : \ell^2(V(T)) \rightarrow \ell^2(V(T))$. For each vertex $v \in V(T)$ and integer $k \geq 0$, denote $C_T(2k, v)$ the number of closed length- $2k$ walks starting at v as usual, so that $C_T(2k, v) = \langle \delta_v, A_T^{2k} \delta_v \rangle$. Define $M_{2k} = \max_{v \in V(T)} C_T(2k, v)$. We need to show that the exponential growth rate of these quantities recovers the spectral radius $\rho(T)$.

Lemma 3.6 (Spectral Radius from Closed Walks). *With notations as above, $\lim_{k \rightarrow \infty} (M_{2k})^{1/2k} = \rho(T)$.*

Proof. Showing upper bound is easy. For every v ,

$$C_T(2k, v) = \langle \delta_v, A_T^{2k} \delta_v \rangle \leq \|A_T^{2k}\|_{\text{op}} \|\delta_v\|^2 = \rho(T)^{2k}. \quad (14)$$

Thus $\limsup_{k \rightarrow \infty} (M_{2k})^{1/2k} \leq \rho(T)$.

For lower bound, by the spectral theorem for bounded self-adjoint operators, there is a projection-valued measure E on \mathbb{R} with support equal to the spectrum of A_T , and $A_T = \int \lambda dE(\lambda)$. For each vertex v , define the scalar spectral measure

$$\mu_v(B) = \langle E(B)\delta_v, \delta_v \rangle, \quad \text{for } B \subset \mathbb{R} \text{ Borel.} \quad (15)$$

Then μ_v is a probability measure and

$$C_T(2k, v) = \langle \delta_v, A_T^{2k} \delta_v \rangle = \int \lambda^{2k} d\mu_v(\lambda). \quad (16)$$

Let $\beta_v = \sup\{|\lambda| : \lambda \in \text{supp}(\mu_v)\}$. We claim first that $\lim_{k \rightarrow \infty} (C_T(2k, v))^{1/2k} = \beta_v$ for fixed v ; the proof is twofold:

- Since $|\lambda| \leq \beta_v$ on the support of μ_v ,

$$C_T(2k, v) \leq \int |\lambda|^{2k} d\mu_v(\lambda) \leq \beta_v^{2k} \quad (17)$$

so that $\limsup_{k \rightarrow \infty} C_T(2k, v)^{1/2k} \leq \beta_v$.

- Conversely, fix $\epsilon > 0$. By definition of β_v , the set $S_\epsilon = \{|\lambda| \geq \beta_v - \epsilon\}$ has positive μ_v -measure, say $\mu_v(S_\epsilon) = \alpha > 0$. For all k ,

$$C_T(2k, v) = \int \lambda^{2k} d\mu_v(\lambda) \geq \int_{S_\epsilon} |\lambda|^{2k} d\mu_v(\lambda) \geq \alpha(\beta_v - \epsilon)^{2k}. \quad (18)$$

Hence, $\liminf_{k \rightarrow \infty} C_T(2k, v)^{1/2k} \geq \beta_v - \epsilon$.

Now, by definition, $\rho(T) = \sup\{|\lambda| : \lambda \in \text{spec}(A_T)\}$. Moreover, the spectrum of A_T is the closure of the union of the supports of the measures μ_v , so $\rho(T) = \sup_v \beta_v$. With abuse of notation, pick any $\epsilon > 0$. Choose a vertex v with $\beta_v > \rho(T) - \epsilon$. Then,

$$\lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} = \beta_v \geq \rho(T) - \epsilon, \quad (19)$$

so that for sufficiently large k , $M_{2k}^{1/2k} \geq C_T(2k, v)^{1/2k} \geq \rho(T) - 2\epsilon$. Combining this with the upper bound and letting $\epsilon \rightarrow 0$ gives the claim. \square

3.4 Proof of the Greenberg-Lubotzky Theorem

We now assemble the ingredients to prove Theorem 3.2.

Proof of Theorem 3.2. Let $\{G_i\}$ be as in the statement: each G_i is finite, connected, and there is a covering map $\pi_i : T \rightarrow G_i$. Let A_i be the adjacency matrix of G_i and $\lambda(G_i)$ its largest nontrivial eigenvalue in absolute value.

Because T has bounded degrees (since it covers a finite graph), the number of vertices in a ball of radius r in T grows at most exponentially in r . For each i , let Δ_i be the diameter of G_i . Since G_i is the quotient of T with $|V(G_i)| \rightarrow \infty$, we necessarily have $\Delta_i \rightarrow \infty$. Otherwise, all G_i would sit inside a fixed ball of T and could not be infinitely many distinct finite quotients.

Fix i and abbreviate G_i, A_i, Δ_i as G, A, Δ , respectively. Choose vertices $s, t \in V(G)$ with $d_G(s, t) = \Delta$. Set $k = k_i = \lfloor (\Delta - 1)/2 \rfloor$ so that $2k < \Delta$. By Lemma 3.4 we have

$$\lambda(G)^{2k} \geq \frac{C_G(2k, s) + C_G(2k, t)}{2}. \quad (20)$$

We next compare closed paths in G and in T . For each of s, t , choose a lift $\tilde{s}, \tilde{t} \in V(T)$ with $\pi_i(\tilde{s}) = s, \pi_i(\tilde{t}) = t$. By Lemma 3.5, $C_G(2k, s) \geq C_T(2k, \tilde{s})$ and $C_G(2k, t) \geq C_T(2k, \tilde{t})$, so

$$\lambda(G)^{2k} \geq \frac{C_T(2k, \tilde{s}) + C_T(2k, \tilde{t})}{2} \geq \frac{M_{2k}}{2} \quad (21)$$

where $M_{2k} = \max_{v \in V(T)} C_T(2k, v)$. Taking the $(2k)^{\text{th}}$ roots, we find $\lambda(G) \geq (1/2)^{1/2k} M_{2k}^{1/2k}$.

Now let $i \rightarrow \infty$. Then $\Delta_i \rightarrow \infty$ and hence $k_i \rightarrow \infty$. From Lemma 3.6 we know that $\lim_{k \rightarrow \infty} M_{2k}^{1/2k} = \rho(T)$, and clearly $(1/2)^{1/2k} \rightarrow 1$ as $k \rightarrow \infty$. Therefore, given any $\epsilon > 0$, we can choose K sufficiently large such that for all $k \geq K$,

$$M_{2k}^{1/2k} \geq \rho(T) - \frac{\epsilon}{2}, \quad \left(\frac{1}{2}\right)^{1/2k} \geq 1 - \frac{\epsilon}{2\rho(T)}. \quad (22)$$

For sufficiently large i we also have $k_i \geq K$, so

$$\lambda(G_i) \geq \left(1 - \frac{\epsilon}{2\rho(T)}\right) \left(\rho(T) - \frac{\epsilon}{2}\right) \geq \rho(T) - \epsilon. \quad (23)$$

Since $\epsilon > 0$ was arbitrarily, this shows

$$\liminf_{i \rightarrow \infty} \lambda(G_i) \geq \rho(T), \quad (24)$$

completing the proof of the Greenberg-Lubotzky theorem. \square