**590.05 Expanders and High-dimensional Expanders** **Fall 2025**

# Lecture 1: Introduction to Expanders

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## 1.1 Motivations of Expanders

Informally, expanders are graphs without very sparse cuts. The formal definition is given below.

**Definition 1.1.** *An undirected graph $G = (V, E)$ is a one-sided $\lambda$-expander if its normalized adjacency matrix $\overline{A}$ satisfies that $\lambda_2(\overline{A}) \le \lambda$. $G$ is a two-sided $\lambda$-expander if $|\lambda_2|(\overline{A}) \le \lambda$ where $|\lambda_2|(\overline{A})$ is the second largest eigenvalue of $\overline{A}$ in absolute value.*

Despite the simple definition, expanders are really useful objects in various areas in computer science, partially because the spectral condition implies several nice structural properties in the graph. We shall examine a couple of such properties later. Now to demonstrate the range of different problems that can be solved with using , we give a couple examples here. After the first two weeks of the lectures you will be able to solve them yourself.

**Problem 1.2** (How to draw graphs in 2D?). *Given a graph $G = (V, E)$ how do we draw the graph in $\mathbb{R}^2$ nicely? Here by nice, we would want vertices that are adjacent in the graph to be close to each other in Eculidean distance. More formally we phrase the requirement to be that find a map from $M : V \to \mathbb{R}^2$ such that $\sum_{v \in V} \|M(v)\|_2^2 = 1$ and the sum of squared distances $\sum_{uv \in E} \|M(u) - M(v)\|_2^2$ is minimized.*

**Problem 1.3** (The sparest cut problem). *In a graph $G = (V, E)$, a cut induced by $S \subseteq V$ is the collection of edges (denoted by $\partial S$, the "boundary" of $S$) between $S$ and its complement. A cut induced by $S$ ($|S| \le |V|/2$) is sparse if only a small fraction of edges adjacent to $S$ are in the cut $\partial S$. The computational problem is given a graph $G$ can we efficiently find the sparsest cut in $G$? Note that the exact problem is* NP*-hard, so we just need a good approximation algorithm.*

**Problem 1.4** (Good linear Codes). *A linear code $C_n$ is a subspace in some vector space ${\mathbb{F}_p}^n$. A main application of codes is data transmission across noisy channels. A couple relevant parameters include the rate of the code $r(C_n) = \frac{\dim(C_n)}{n}$, the distance of the code $\delta(C_n) = \min_{c \in C_n \setminus \vec{0}} \mathrm{Hamming}(c)$, and the running time $t(C_n)$ of the decoding algorithm that takes in a corrupted codeword and outputs the codeword closest to the input.*

*The question is: can we construct an infinite family of linear codes $\{C_n\}_n$ such that there exists $r_0, \delta_0 > 0$ and $T(n) \in O(n)$ satisfying that for all $C_n$, $r(C_n) \ge r_0$, $\delta(C_n) \ge \delta_0$, and $C_n$ has a decoding algorithm running in time $T(n)$?*

**Problem 1.5.** *Derandomizing* BPP *Recall that* BPP *is the class of languages that can be solved by probabilistic polynomial time Turing machines with bounded, two-sided errors. The common belief is that* BPP = P, *so an important question is to understand how much randomness is actually needed for these problems. While proving the holy grail that $O(\log n)$ bits of randomness suffices, there have been many successes in showing that it is possible to reduce the amount randomness. Here is a question along this line.*

*Given an algorithm $A$ for a* BPP *language $L$ that uses $m$ random bits and achieves $< \frac{1}{3}$ two-sided errors, one can reduce the errors to any $\varepsilon > 0$ by repeating the algorithm $\log(1/\varepsilon)$ times and output the majority vote of the outputs. However this approach uses $m \log(1/\varepsilon)$ random bits. Can we achieve the same error bound while using only $m + O(\log(1/\varepsilon))$ bits of randomness?*

The usefulness of expanders root in the fact that the spectral definition surprisingly implies an array of combinatorial, geometric, and probabilistic properties. We next carefully examine these properties.

## 1.2 Notations and Facts

In this section we introduce notations and facts about graphs and matrix spectra.

Give a graph $G = (V, E)$ on $n$ vertices, we use $D$ to denote the diagonal matrix of vertex degrees, $A$ to denote the adjacency matrix, and $L$ to denote the unnormalized graph Laplacian $D - A$.

We also define the normalized adjacency matrix to be $\overline{A} = D^{-1/2}AD^{-1/2}$, and the normalized graph Laplacian to be $\overline{L} = D^{-1/2}LD^{-1/2} = I - \overline{A}$.

Let the spectrum of $\overline{A}$ be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and the corresponding eigenvectors be $v_1, v_2 \ldots, v_n$. Let the spectrum of $L$ be $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and the corresponding eigenvectors be $u_1, u_2 \ldots, u_n$. Here are some facts about the spectra and eigenspaces of $\overline{A}$ and $\overline{L}$.

**Fact 1.6.** *1. $\lambda_1 = 1$ and $v_1 \propto D^{1/2}\vec{1}$, $\lambda_n \geq -1$. 2. for all $i \in [n]$, $\gamma_i = 1 - \lambda_i$ and $u_i = v_i$.*

**Fact 1.7.** *For a matrix $M$ with spectrum $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and corresponding eigenvectors $u_1, u_2, \ldots, u_n$, its $i$-th smallest eigenvalue can be equivalently defined using Raleigh quotient as*

$$\lambda_i = \min_{x \perp \mathrm{Span}(u_1, \ldots, u_{i-1})} \frac{x^\top M x}{\langle x, x \rangle}.$$

**Fact 1.8.** *For an unweighted graph $G$, the quadratic forms of $L$ and $\overline{L}$ can be written as*

$$\langle f, Lf \rangle = \sum_{ab \in E} (f(a) - f(b))^2, \ \langle f, \overline{L}f \rangle = \sum_{ab \in E} \left( \frac{1}{\sqrt{d_a}} f(a) - \frac{1}{\sqrt{d_b}} f(b) \right)^2.$$

*where $d_a, d_b$ are the degrees of $a$ and $b$ inside $G$.*

## 1.3 Geometric property: conductance

The first implication of graph expansion is that for every set $S \subset V$ that is not too large at least $\frac{\gamma_2}{2}$-fraction of its adjacent edges are in the cut $\partial S$.

**Definition 1.9** (Conductance)**.** *Given a graph $G = (V, E)$ and a set $S \subseteq V$, we define the conductance of the set to be*

$$\phi(S) = \frac{|\partial S|}{\min(\mathrm{vol}(S), \mathrm{vol}(\overline{S}))},$$

*where* $\mathrm{vol}(S) = \sum_{a \in S} d_a$ *and* $\overline{S} = V \setminus S$.

Just like the original physics notion that measures how easily an electric current flows through a material, here conductance measures how easily a random walk in $G$ starting from a set $S$ escapes $S$.

**Definition 1.10** (Graph conductance)**.** *The conductance of a graph $G = (V, E)$ is*

$$\phi(G) = \min_{S \subseteq V} \phi(S).$$

The following classic result captures the connections between graph expansion and conductance.

**Lemma 1.11.** *(Cheeger's inequality) Let $G$ be a graph with Laplacian spectrum $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$. Then*

$$\frac{\gamma_2}{2} \le \phi(G) \le \sqrt{2\gamma_2}.$$

One consequence of the lower bound on $\phi(G)$ is that if $\gamma_2$ is bounded away from 0, then it is hard to disconnect $G$. In particular, even after removing $\varepsilon$-fraction of the edges, $G$ still has a large connected component of size $\ge (1 - \varepsilon/2\phi(G))n$. The proof is left as an exercise.

We first prove the lower bound, and postpone the proof of the upper bound till later.

*Proof of the lower bound side of Cheeger's inequality.* Let $x = D^{1/2}\vec{1}$ denote the smallest eigenvector $L$ of $G$.

For any $S \subseteq V$ such that $\mathrm{vol}(S) \le \mathrm{vol}(\overline{S})$, let $\vec{1}_S$ be the indicator vector of $S$ and

$$f = D^{1/2}\vec{1}_S - \left\langle \vec{1}_S, D\vec{1} \right\rangle \cdot x / \|x\|^2 .$$

Note that $\langle f, x\rangle = \left\langle \vec{1}_S, D\vec{1} \right\rangle - \left\langle \vec{1}_S, D\vec{1} \right\rangle = 0$ so $f \perp x$. Compute

$$\langle f, f\rangle = \left\langle \vec{1}_S, D\vec{1}_S \right\rangle + \left\langle \vec{1}_S, D\vec{1} \right\rangle^2 / \|x\|^2 - 2\left\langle \vec{1}_S, D\vec{1} \right\rangle^2 / \|x\|^2 = \mathrm{vol}(S) - \mathrm{vol}(S)^2/\mathrm{vol}(V),$$

and

$$f^\top L f = \left\langle \vec{1}_S, (D-A)\vec{1}_S \right\rangle = \mathrm{vol}(S) - 2\,|E(S,S)| = \left|E(S,\overline{S})\right|$$

Recall that by Fact 1.7 every $f \perp x$ satisfies that $\gamma_2 \le \frac{z^\top A z}{\langle z, z\rangle}$.

$$\gamma_2 \le \frac{f^\top L f}{\langle f, f\rangle} = \frac{\left|E(S,\overline{S})\right|}{\mathrm{vol}(S) - \mathrm{vol}(S)^2/\mathrm{vol}(V)} \le \frac{\left|E(S,\overline{S})\right|}{\mathrm{vol}(S)/2} = 2\phi(S).$$

Here the second inequality follows from the condition that $\mathrm{vol}(S) \le \mathrm{vol}(\overline{S})$. Since this holds for every set $S$ satisfying this condition, we can conclude that

$$\phi(G) = \min_{S \mid \mathrm{vol}(S) \le \mathrm{vol}(\overline{S})} \phi(S) \ge \frac{\gamma_2}{2}.$$

□

# Lecture 2: Applications of Expanders

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## 2.1 Pseduorandom property: expander mixing lemma

Intuitively, a graph is *pseudorandom* if its edge distribution closely resembles that of a truly random graph.

Consider a random $d$-regular graph $H = (V, \mathcal{E})$ on $n$ vertices. For two subsets $S, T \subseteq V$, the expected number of edges between $S$ and $T$ is

$$\mathbb{E}[\,|\mathcal{E}(S,T)|\,] \;=\; \frac{d}{n}\,|S|\,|T|.$$

Motivated by this, we say that a $d$-regular graph $G = (V, E)$ is *pseudorandom* if, for every pair of subsets $S, T \subseteq V$, the number of edges between them satisfies

$$|E(S,T)| \;\approx\; \frac{d}{n}\,|S|\,|T|.$$

The following lemma shows that $d$-regular expander graphs are pseudorandom in exactly this sense: they have nearly uniform edge distribution between all vertex subsets.

**Lemma 2.1** (Expander mixing lemma)**.** *Let $G = (V, E)$ be a $d$-regular graph on $n$ vertices. If $G$ is a* ***two-sided*** *$\lambda$-expander, then for any subsets $S, T \subseteq V$,*

$$\left|\; |E(S,T)| - \frac{d}{n}|S||T| \;\right| \;\leq\; \lambda d \sqrt{|S||T|}.$$

*Proof.* Let $\vec{1}_S, \vec{1}_T \in \mathbb{R}^n$ be the indicator vectors of subsets $S, T \subseteq V$. Then

$$|E(S,T)| \;=\; d\,\langle \vec{1}_S,\, \overline{A}\,\vec{1}_T\rangle,$$

where $\overline{A} = A/d$ is the normalized adjacency matrix.

Now decompose $\vec{1}_S$ and $\vec{1}_T$ into components parallel and orthogonal to $\vec{1}$:

$$f_S \;=\; \vec{1}_S - \frac{|S|}{n}\vec{1}, \qquad f_T \;=\; \vec{1}_T - \frac{|T|}{n}\vec{1}.$$

Since $\vec{1}$ is an eigenvector of $\overline{A}$ with eigenvalue 1, we obtain

$$|E(S,T)| = d \cdot \left\langle \frac{|S|}{n}\vec{1},\, \overline{A}\,\frac{|T|}{n}\vec{1} \right\rangle \;+\; d \cdot \langle f_S, \overline{A} f_T \rangle.$$

The first term simplifies to

$$\frac{d}{n}\,|S|\,|T|,$$

which is exactly the expected edge count in a random $d$-regular graph.

For the second term, note that $\overline{A}$ acts on the subspace $\vec{1}^{\perp}$ with operator norm at most $\lambda$, the second largest eigenvalue in absolute value. Hence

$$\left| d\langle f_S, \overline{A} f_T\rangle \right| \;\leq\; \lambda d \, \|f_S\| \, \|f_T\|.$$

Since $\|f_S\|^2 \leq |S|$ and $\|f_T\|^2 \leq |T|$, it follows that

$$\left| d\langle f_S, \overline{A} f_T\rangle \right| \;\leq\; \lambda d \sqrt{|S||T|}.$$

Combining both terms gives

$$\left| |E(S,T)| - \tfrac{d}{n}|S||T| \right| \;\leq\; \lambda d \sqrt{|S||T|}.$$

□

The classic Expander Mixing Lemma applies to $d$-regular graphs. It also extends naturally to irregular graphs. We state the generalized version below and leave the proof as an exercise.

**Lemma 2.2.** *Let $G = (V, E)$ be a graph on $n$ vertices. If $G$ is a two-sided $\lambda$-expander, then for any subsets $S, T \subseteq V$,*

$$\left| \, |E(S,T)| - \frac{\mathrm{vol}(S) \cdot \mathrm{vol}(T)}{\mathrm{vol}(V)} \, \right| \;\leq\; \lambda \sqrt{\mathrm{vol}(S)\mathrm{vol}(T)}.$$

The expander mixing lemma stated above relies on two-sided expansion. In fact, one-sided expansion is insufficient. As a concrete example, consider bipartite graphs: such graphs cannot be two-sided expanders since their normalized adjacency matrix always has an eigenvalue $-1$. Indeed, for a one-sided bipartite graph $G = (U, V, E)$ with bipartition $U$ and $V$, if we take $S, T \subseteq U$ (or both in $V$), then $|E(S,T)| = 0$. Thus the standard expander mixing lemma fails in this setting. Nevertheless, there is a natural version of the expander mixing lemma tailored for bipartite graphs, which controls the number of edges between subsets of $U$ and $V$.

**Lemma 2.3** (Expander Mixing Lemma for Bipartite Graphs)**.** *Let $G = (U, V, E)$ be a bipartite graph. If $G$ is a one-sided $\lambda$-expander, then for any subsets $S \subseteq U$ and $T \subseteq V$,*

$$\left| |E(S,T)| - \frac{\mathrm{vol}(S) \cdot \mathrm{vol}(T)}{\mathrm{vol}(V)} \right| \leq \lambda \sqrt{\mathrm{vol}(S)\mathrm{vol}(T)}.$$

The proof is analogous to the proof of the two-sided case, except that instead of working with $\overline{A}$, we work with the normalized bi-adjacency matrix $B$ of $G$. Observe that

$$\overline{A} = \begin{bmatrix} 0 & B \\ B^{\top} & 0 \end{bmatrix}.$$

So $B$'s singular values $1 = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are precisely $\sigma_i = |\lambda_i|$. In particular $\sigma_2 \leq \lambda$. The remainder of the argument proceeds as in the two-sided case.

## 2.2 Probabilistic property: rapid mixing

A *random walk* on a graph $G = (V, E)$ is the Markov chain defined as: from a vertex $u \in V$, the walk moves to a uniformly random neighbor $v$ of $u$. The transition matrix of this process is

$$P = AD^{-1}.$$

Thus $P$ is column-stochastic. The stationary distribution $\pi$ is given by

$$\pi(v) = \frac{d_v}{\text{vol}(V)},$$

and satisfies the invariance property

$$P\pi = \pi.$$

Note that if we start with an initial distribution $\nu$ on the vertices, then after one step of the random walk the distribution becomes $P\nu$.

**Definition 2.4.** *For two distributions $\pi, \nu$ over a finite set $V$, their* total variation distance *is*

$$\|\pi - \nu\|_{TV} = \frac{1}{2} \sum_{u \in V} \big|\pi(u) - \nu(u)\big|.$$

We say that a random walk *mixes* if for every initial distribution $\nu$, the sequence $\{P^t \nu\}_{t \geq 0}$ converges to the stationary distribution $\pi$ in total variation distance, i.e.

$$\lim_{t \to \infty} \big\|P^t \nu - \pi\big\|_{TV} \; = \; 0.$$

Moreover, the walk is said to *mix rapidly* if this distance decays exponentially fast with $t$.

Intuitively, in a poorly connected graph such as a path, the random walk can remain trapped in one region for many steps, leading to slow mixing. By contrast, in an expander graph every vertex set has many edges leaving it, so the walk escapes quickly from any region, leading to fast convergence. This intuition is formalized in the following theorem.

**Lemma 2.5** (Rapid mixing on expanders)**.** *Let $G = (V, E)$ be a graph. If $G$ is a* ***two-sided*** *$\lambda$-expander, then for any initial distribution $\nu$ on $V$, after $t$ steps of the random walk,*

$$\big\|P^t \nu - \pi\big\|_{TV} \leq \frac{1}{2} \sqrt{\frac{1}{\pi_{\min}}} \lambda^t,$$

*where $\pi_{\min} = \min_{v \in V} \pi(v)$ is the smallest probability mass in the stationary distribution.*

From the lemma it follows that if the vertex degrees are bounded by a constant, then the total variation distance decays exponentially with $t$, and hence the random walk exhibits rapid mixing.

**Corollary 2.6.** *If a two-sided $\lambda$-expander has bounded degree, then the mixing time to reach $\varepsilon$-total variation closeness to the stationary distribution is*

$$t_{\text{mix}}(\varepsilon) \; = \; O\biggl(\frac{\log(1/\varepsilon) + \log(1/\pi_{\min})}{1 - \lambda}\biggr).$$

Now we proceed to prove the rapid mixing lemma.

*Proof.* Let the eigenvalues of the normalized adjacency matrix $\overline{A}$ be $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, with the top eigenvector $v_1 = \frac{1}{\sqrt{\text{vol}(V)}} D^{1/2} \vec{1}$.

Note that the random walk operator $P$ is similar to $\overline{A}$

$$P = AD^{-1} = D^{1/2} \overline{A} D^{-1/2},$$

and so has the same eigenvalues. Furthermore, the $i$-th eigenvectors $v_i$ of $\overline{A}$ and $u_i$ of $P$ satisfy $u_i = D^{1/2} v_i$. In particular $u_1 = \frac{1}{\sqrt{\mathrm{vol}(V)}} D\vec{1} = \sqrt{\mathrm{vol}(V)} \cdot \pi$.

To prove the statement for any initial distribution $\nu$, it suffices to prove it for distributions of the form $p_0 = \vec{1}_v$ (the indicator of a vertex) where $v \in V$ and then use the convexity of TV distance to conclude the claim. So next we prove the claim for such distributions.

**Step 1: Moving from $P$ to $\overline{A}$.** Let $p_0 = \vec{1}_v$ be the distribution of a random walk starting from a vertex $v \in V$. Consider the weighted vector

$$q_0 = D^{-1/2} p_0 \in \mathbb{R}^n.$$

Let $p_t = P^t p_0$, we have

$$q_t = D^{-1/2} p_t = \overline{A}^t q_0.$$

**Step 2: Projection onto eigenbasis.** Expand $q_0$ in the orthonormal eigenbasis $\{v_i\}$ of $\overline{A}$:

$$q_0 = \sum_{i=1}^{n} \langle q_0, v_i \rangle \, v_i.$$

Then

$$q_t = \overline{A}^t q_0 = \sum_{i=1}^{n} \langle q_0, v_i \rangle \, \lambda_i^t v_i.$$

**Step 3: Bounding the contributions from $\mathrm{Span}(v_1)$ and from $v_1^{\perp}$.** Note that the first term

$$\langle q_0, v_1 \rangle \cdot 1^t \cdot v_1 = \frac{\left\langle \vec{1}_v, \vec{1} \right\rangle}{\sqrt{\mathrm{vol}(V)}} \cdot v_1 = \frac{1}{\sqrt{\mathrm{vol}(V)}} D^{-1/2} u_1 = D^{-1/2} \pi.$$

The rest of the terms

$$q_t - \langle q_0, v_1 \rangle \cdot 1^t \cdot v_1 = \sum_{i=2}^{n} \lambda_i^t \langle q_0, v_i \rangle \, v_i.$$

Its norm satisfies

$$\| q_t - \langle q_0, v_1 \rangle \, v_1 \| \le \lambda^t \, \| q_0 \|.$$

Since $q_0 = D^{-1/2} \vec{1}_v$, we have $\|q_0\| = 1/\sqrt{d_v}$.

**Step 4: Connect back to TV distance.** Because $p_t = D^{1/2} q_t$, the TV distance satisfies

$$\| p_t - \pi \|_{TV} = \frac{1}{2} \| D^{1/2} (q_t - D^{-1/2} \pi) \|_1 \;\le\; \frac{1}{2} \sqrt{\sum_{u \in V} d_u} \cdot \| q_t - \langle q_0, v_1 \rangle \, v_1 \|,$$

where the last step uses Cauchy–Schwarz. Combining with Step 3 gives

$$\| p_t - \pi \|_{TV} \;\le\; \frac{1}{2} \sqrt{\frac{1}{\pi(v)}} \, \lambda^t.$$

Finally, observe that $\pi_{\min} \le \pi(v)$, so we conclude that

$$\left\| P^t p_0 - \pi \right\|_{TV} \;\le\; \frac{1}{2}\sqrt{\frac{1}{\pi_{\min}}}\,\lambda^t.$$

$\square$

# Lecture 3: Applications of Expanders

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We now turn to applications of these structural properties of expander graphs. In particular, we will see how expanders play a central role in approximation algorithms, coding theory, and derandomization.

## 3.1 Application 1: Cheeger's inequality and the sparest cut algorithm

The *sparsest cut problem* over a graph $G$ asks to find a subset $S \subseteq$ that minimize the edge expansion $\phi(S)$. The exact problem is NP-hard, but the following spectral algorithm gives a $2/\sqrt{\phi(G)}$-approximation.

**Algorithm 1** Spectral Algorithm for Sparsest Cut

**Require:** Graph $G = (V, E)$
**Ensure:** A cut $(S, \overline{S})$ with small edge expansion

Compute the matrix $M = I - D^{-1}A$.
Find the eigenvector $f$ corresponding to the second smallest eigenvalue of $M$.
Sort the vertices $v_1, v_2, \ldots, v_n$ such that $f(v_1) \le f(v_2) \le \cdots \le f(v_n)$.
**for** $k = 1$ to $n-1$ **do**
  Let $S_k = \{v_1, v_2, \ldots, v_k\}$.
  Compute conductance:
$$\phi(S_k) = \frac{|\partial S_k|}{\min\{\mathrm{vol}(S_k),\, \mathrm{vol}(\overline{S_k})\}}.$$
**end for**
Output the cut $S_k$ with the minimum conductance $\phi(S_k)$.

The analysis of this algorithm is closely related to the proof of the upper bound $\phi(G) \le \sqrt{2\gamma_2}$ in the Cheeger's inequality. We first prove the correctness of this algorithm and then use it to deduce the upper bound side of Cheeger's inequality.

**Theorem 3.1.** *Let $G$ be a graph whose normalized Laplacian has second smallest eigenvalue $\gamma_2$. Then the spectral algorithm outputs a cut $S$ such that*
$$\phi(S) \le \sqrt{2\gamma_2}.$$

Since $\phi(S)$ is an upper bound on the graph conductance, we immediately derive the upper bound side of the Cheeger's inequality as a corollary.

**Corollary 3.2.** *Let $G$ be a graph with Laplacian spectrum $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$.Then*
$$\phi(G) \le \sqrt{2\gamma_2}.$$

*Proof.* Note that $M = D^{-1/2}\overline{L}D^{1/2}$ is similar to $\overline{L}$. Let $g$ be the eigenvector of $\overline{L}$ corresponding to the second smallest eigenvalue $\gamma_2$. Then $f = D^{-1/2}g$ is the eigenvector of $M$ corresponding to $\gamma_2$, and thereby $\langle f, Lf \rangle = \langle g, \bar{L}g \rangle$ and $\langle f, Df \rangle = \langle g, g \rangle$.

**Step 1: Sweep sets.** Center $f$ to be $\bar{f} = f - m \cdot \vec{1}$ such that $\mathrm{vol}(S_m), \mathrm{vol}(\bar{S}_m) \le \mathrm{vol}(V)/2$. Note that $\langle \bar{f}, L\bar{f} \rangle = \langle f, Lf \rangle \langle g, \bar{L}g \rangle$ and $\langle \bar{f}, D\bar{f} \rangle = \langle f, Df \rangle + m^2 \cdot \mathrm{vol}(V) = \langle g, g \rangle + m^2 \cdot \mathrm{vol}(V)$.

Sort the vertices $v_1, v_2, \ldots, v_n$ such that

$$a = \bar{f}(v_1) \le \bar{f}(v_2) \le \cdots \le \bar{f}(v_n) = b.$$

For thresholds $t \in [a, b]$, define the set

$$S_t = \{v \in V : \bar{f}(v) < t\}.$$

**Step 2: Averaging argument.** In this step, we define a distribution $\mu$ over $[a, b]$ such that

$$\frac{\mathrm{E}_{t \sim \mu}[|\partial S_t|]}{\mathrm{E}_{t \sim \mu}[\min \mathrm{vol}(S_t), \mathrm{vol}(\overline{S_t})]} \le \sqrt{2\gamma_2}. \tag{3.1}$$

The distribution is defined to be $\mu(t) = \frac{2|t|}{a^2+b^2}$. We can verify that $\int_a^b \mu(t) = \frac{1}{a^2+b^2} \cdot \left(\frac{2a^2}{2} + \frac{2b^2}{2}\right) = 1$ this is a valid probability distribution.

We next compute the two expectations.

$$\begin{aligned}
\mathrm{E}_{t \sim \mu}[|\partial S_t|] &= \sum_{\{v_i, v_j\} \in E} \Pr_{t \sim \mu}[\{v_i, v_j\} \in \partial S_t] \\
&= \sum_{\{v_i, v_j\} \in E} \Pr_{t \sim \mu}[t \in [\bar{f}(v_i), \bar{f}(v_j)]] \\
&= \sum_{\{v_i, v_j\} \in E} \left(\mathrm{sgn}(\bar{f}(v_j))\bar{f}(v_j)^2 - \mathrm{sgn}(\bar{f}(v_i))\bar{f}(v_i)^2\right) \\
&\le \sum_{\{v_i, v_j\} \in E} \left(\bar{f}(v_j) - \bar{f}(v_i)\right) \cdot \left|\bar{f}(v_i) + \bar{f}(v_j)\right| \\
&\le \sqrt{\langle \bar{f}, (D - A)\bar{f} \rangle} \cdot \sqrt{\sum_{\{v_i, v_j\} \in E} (\bar{f}(v_i) + \bar{f}(v_j))^2} \\
&\le \sqrt{\langle g, \bar{L}g \rangle} \cdot \sqrt{2(\langle g, g \rangle + m^2 \cdot \mathrm{vol}(V))}
\end{aligned}$$

Recall that by construction

$$\min\{\mathrm{vol}(S_t), \mathrm{vol}(\overline{S_t})\} = \begin{cases} \mathrm{vol}(S_t) & t \le 0 \\ \mathrm{vol}(\overline{S_t}) & t > 0 \end{cases},$$

and

$$\begin{aligned}
\mathrm{E}_{t \sim \mu}[\min\{\mathrm{vol}(S_t), \mathrm{vol}(\overline{S_t})\}] &= \sum_{v_i \in V, \bar{f}(v_i) < 0} d_{v_i} \cdot \Pr_{t \sim \mu}[t \in [\bar{f}(v_i), 0)] + \sum_{v_i \in V, \bar{f}(v_i) \ge 0} d_{v_i} \cdot \Pr_{t \sim \mu}[t \in [0, \bar{f}(v_i)]] \\
&= \sum_{v_i \in V} d_{v_i} \bar{f}(v_i)^2 \\
&= \langle g, g \rangle + m^2 \cdot \mathrm{vol}(V)
\end{aligned}$$

The Raleigh quotient definition of $\gamma_2$ implies that

$$\frac{\mathrm{E}_{t\sim\mu}[|\partial S_t|]}{\mathrm{E}_{t\sim\mu}[\min \mathrm{vol}(S_t), \mathrm{vol}(\overline{S_t})]} \leq \sqrt{\frac{2\left\langle g, \overline{L}g \right\rangle}{\langle g, g\rangle + m^2 \cdot \mathrm{vol}(V)}} = \sqrt{2\gamma_2}.$$

Therefore there exists some set $S_t$ such that $\frac{|\partial S_t|}{\mathrm{vol}(S_t)} \leq \sqrt{2\gamma_2}$. □

## 3.2 Application 2: expansion and linear codes

In this section we first construct a class of linear codes from bipartite graphs, and then show that when the underlying graph is a good bipartite vertex expander, the resulting codes have constant rate, constant relative distance, and a linear-time decoding algorithm. Finally, we discuss the relationship between bipartite vertex expanders and general bipartite expanders.

**Definition 3.3.** *Let $G = (L \cup R, E)$ be a bipartite expander graph, where*

- *$L$ is the set of $n$* variable nodes *(codeword positions),*
- *$R$ is the set of $m$* constraint nodes *(parity checks),*
- *each vertex in $L$ has constant degree $d$,*

*Define the code $C(G)$ over $\mathbb{F}_2$ to be*

$$C(G) \;=\; \Big\{ x \in \mathbb{F}_2^n \;:\; \sum_{u \in N(r)} x_u \equiv 0 \pmod 2 \quad \text{for all } r \in R \Big\},$$

*where $N(r)$ denotes the set of neighbors of $r \in R$. Equivalently let $B \in \mathbb{F}_2^{m\times n}$ be the bi-adjacency matrix of $G$. Then $C(G) = \mathrm{Ker}(B)$ is the right kernel of $B$.*

Later in the course we will see a generalization of the graph codes called Tanner codes where instead of requiring $\sum_{u\in N(r)} x_u \equiv 0 \pmod 2$ for all $r \in R$, it requires that $x|_{N(r)} \in C_r$ to be in some local code $C_r$ for every $r \in R$.

Now we state the properties of the graph code $C(G)$ when $G$ has the following vertex expansion property.

**Theorem 3.4.** *Let $G$ be as in Definition 3.3. If there exist $\alpha \in (0, 1]$ and $\beta \in (\frac{3}{4}, 1]$ such that for every $S \subseteq L$ where $|S| \leq \alpha n$, $|N(S)| \geq \beta d\,|S|$, then the code $C(G)$ satisfies that*

1. *$r(C(G)) \geq \frac{n-m}{n}$;*
2. *$\delta(C(G)) > \alpha$;*
3. *there exists an algorithm $A$ such that on input $y \in \mathbb{F}_2^n$ such that $\mathrm{Hamming}(y, C) < \frac{\alpha n}{2}$, it outputs the closest codeword $c \in C$ to $y$ and runs in time $\Theta_d(n \cdot \mathrm{Hamming}(y, C))$.*

*Proof.* The proof of the first statement follows from a straightforward dimension-counting argument.

For the second statement, suppose for contradiction that there exists a nonzero codeword $x \in C(G)$ with Hamming weight $\leq \alpha n$. Let $S_x \subseteq L$ denote the support of $x$, i.e. the set of variable nodes corresponding to nonzero coordinates of $x$. Since all constraints $\sum_{u\in N(r)} x_u \equiv 0 \pmod 2$ for $r \in N(S_x)$ are satisfied.

This implies that each $r \in N(S_x)$ is adjacent to at least 2 vertices in $S_x$. Hence $|N(S_x)| \leq \frac{d|S_x|}{2}$. On the other hand, by the vertex expansion property of $G$ we have $|N(S_x)| \geq \frac{3}{4}d\,|S_x|$. The two bounds yield a contradiction, and so $\delta(C(G)) > \alpha$.

We will prove the final statement in the next lecture. □

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# Lecture 4: Applications of Expanders (cont.)

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 4.1 Application 2: expansion and linear codes

We finish the proof by giving a linear-time decoding algorithm.

**Theorem 4.1.** *Let $G = (L \cup R, E)$ be a left $d$-regular bipartite graph with $|L| = n$, $|R| = m$. If there exist $\alpha \in (0, 1]$ and $\beta \in (\frac{3}{4}, 1]$ such that for every $S \subseteq L$ where $|S| \leq \alpha n$, $|N(S)| \geq \beta d\,|S|$, then the code $C(G)$ satisfies that*

1. *$r(C(G)) \geq \frac{n-m}{n}$;*
2. *$\delta(C(G)) > \alpha$;*
3. *there exists an algorithm $A$ such that on input $y \in \mathbb{F}_2^n$ such that $\mathrm{Hamming}(y, C) \leq \frac{\alpha n}{2}$, it outputs the closest codeword $c \in C$ to $y$ and runs in time $\Theta_d(n \cdot \mathrm{Hamming}(y, C))$.*

*Proof of 3.* Let $A$ be the following algorithm.

**Algorithm 1** Sipser–Spielman Linear-Time Decoding (Bit-Flipping Algorithm)

**Require:** Received word $y \in \{0, 1\}^n$, graph code $C(G)$ defined by bipartite graph $G = (L \cup R, E)$
**Ensure:** Decoded codeword $x \in C(G)$ (or failure if too many errors)

```
Initialize x ← y.
while there exists a variable node u ∈ L with more than half of its incident constraints in R violated do
    Flip the bit: x_u ← 1 − x_u.
end while
return x.
```

We begin by noting that if $\mathrm{Hamming}(y, C) \leq \frac{\alpha n}{2}$, then the number of violated constraints in $y$ is at most $\frac{\alpha d n}{2}$.

Moreover, for such inputs $y$, the algorithm $A$ must terminate within $\frac{\alpha d n}{2}$ iterations, since in each step the number of violated constraints decreases by at least one.

We now establish the correctness of $A$ on these inputs in two parts.

**1. The algorithm cannot output a codeword other than the closest one.** Suppose the decoder outputs some $c' \in C(G)$ with $c' \neq c$, where $c$ is the true nearest codeword to $y$. Then $\mathrm{Hamming}(c, c') \geq \delta(C(G))n > \alpha n$. Hence at some intermediate stage the algorithm must produce a string $x$ with $\mathrm{Hamming}(c, x) = \alpha n$. By the vertex expansion property of $G$, the number of constraints violated by such an $x$ exceeds $\frac{\alpha}{2} dn$. This is strictly larger than the number of constraints violated by the input $y$, contradicting the monotonic decrease of violations under $A$. Therefore $A$ cannot output a codeword other than the unique closest codeword $c$.

**2. The algorithm cannot terminate at a non-codeword.** Suppose for contradiction that $A$ halts at a string $x \notin C(G)$. By definition of the algorithm, no variable node $u \in L$ has a majority of its incident constraints violated.

Let $S \subseteq L$ be the set of coordinates where $x$ differs from the true codeword $c$. By Step 1, we know $|S| < \alpha n$, and since $x \neq c$ we have $S \neq \varnothing$. By vertex expansion, $S$ is adjacent to $> \frac{d}{2}|S|$ distinct violated constraints. By averaging, there exists some $u \in S$ adjacent to $> d/2$ violated constraints, contradicting the assumption that the algorithm had halted. Thus $A$ cannot terminate at a non-codeword.

Together, these two claims show that if $y$ is sufficiently close to $C(G)$, the algorithm always halts at the unique closest codeword $c$. □

Bipartite graphs satisfying the conditions in Theorem 4.1 are called (one-sided) $(\alpha, \beta)$-vertex expanders. We can use the expander mixing lemma to show that for certain choices of $\alpha$ and $\beta$, $(\alpha, \beta)$-vertex expanders can be constructed from bipartite expanders.

**Claim 4.2.** *Let $G = (L \cup R, E)$ be a bi-regular one-sided$\lambda$ expander with left degree $d$, $|L| = n$, and $|R| = m$. Then for any $\alpha < \frac{m}{dn}$, $G$ is a $(\alpha, \frac{(1-\alpha dn/m)^2}{\lambda^2 d})$-vertex expander.*

*Proof.* Consider any $S \subseteq L$ of size $|S| \leq \alpha |L|$ and use $T = N(S)$ to denote the set of vertices adjacent to $S$.

By the expander mixing lemma,

$$d \cdot |S| = |E(S,T)| \leq \frac{d\,|S|\,|T|}{m} + \lambda d \sqrt{|S|\,|T|}.$$

Furthermore, since $|T| = |N(S)| \leq \alpha d n$, we derive a lower bound:

$$\sqrt{\frac{|T|}{|S|}} \geq \frac{1 - \alpha dn/m}{\lambda}.$$

We conclude that

$$|T| \geq \frac{(1-\alpha dn/m)^2}{\lambda d} \cdot d\,|S|\,.$$

□

In fact, random left $d$-regular graphs are excellent vertex expanders with high probability. This is captured by the following lemma which we state without a proof.

**Lemma 4.3.** *For all $0 < \beta < 1$ and $m < n$, there exists $d = \Theta\left(\frac{\log(n/m)}{1-\beta}\right)$ such that w.h.p. a random left $d$-regular graph with $|L| = n$ and $|R| = m$ is a $\left(\Theta\left(\frac{(1-\beta)m}{dm}\right), \beta\right)$-vertex expander.*

**Remark 4.4.** *Explicit constructions of such vertex expanders will not be covered in this class. For those who are interested, you can do a final project on writing notes about a how to construct unique-neighbor expanders or lossless expanders.*

*It turns out that even without a unique-neighbor expander, the same idea can be made to work by using a standard bipartite expander together with a different base code $C_0$ that has larger distance.*

## 4.2 Application 3: expander random walk and derandomization

**Theorem 4.5** (Derandomization of BPP via Expander Walks)**.** *Let $A(x, r)$ be a probabilistic polynomial-time algorithm that on input $x \in \{0,1\}^n$ uses $m = \mathrm{poly}(n)$ random bits $r \in \{0,1\}^m$ and errs with probability at most $1/3$. Then for every $\varepsilon > 0$, there exists an explicit algorithm $A'(x)$ that uses only $m + O(\log(1/\varepsilon))$ random bits, runs in polynomial time, and errs with probability at most $\varepsilon$.*

The standard error reduction for BPP algorithms repeats $A(x, r)$ independently $t$ times and takes the majority outcome. By Chernoff bounds, $t = O(\log(1/\varepsilon))$ repetitions suffice to reduce the error probability to at most $\varepsilon$. However, this requires $t \cdot m$ random bits.

To reduce randomness, we replace independence with a random walk on an expander. Let $G$ be an explicit constant-degree expander with vertex set $V = \{0,1\}^m$ (so $|V| = 2^m$). Each vertex corresponds to a random seed. We pick a uniformly random start vertex $r_0 \in V$ and take a random walk of length $t$ to obtain seeds $r_0, r_1, \ldots, r_{t-1}$. We then run $A(x, r_i)$ for each $i$ and output the majority value.

The key property is the *Expander Chernoff Bound*: although $r_0, \ldots, r_{t-1}$ are correlated, the number of "bad" seeds behaves nearly as in the independent case, and concentration still holds. Thus the probability that the majority of runs is incorrect is at most $\varepsilon$, just as with independent sampling.

Finally, the random bits needed are: $m$ bits to choose the starting seed $r_0$, and $O(t) = O(\log(1/\varepsilon))$ bits to specify the random walk. Hence $A'$ uses only $m + O(\log(1/\varepsilon))$ random bits in total, runs in polynomial time, and achieves error probability $\le \varepsilon$.

Next we state and prove the *Expander Chernoff Bound.*

**Theorem 4.6** (Expander Chernoff Bound)**.** *Let $G = (V, E)$ be a regular graph on $|V| = n$ and a two-sided $\lambda$-expander. Let $B \subseteq V$ be any "bad" set of vertices with density $p = |B|/n$. Suppose we take a random walk $v_0, v_1, \ldots, v_{t-1}$ of length $t$ on $G$, starting from the uniform distribution on $V$. Then for any $\delta > 0$,*

$$\Pr_{v_0,\ldots,v_{t-1}}\left[\frac{1}{t}\sum_{i=0}^{t-1} \mathbf{1}_{\{v_i \in B\}} \;\ge\; p + \lambda + \delta\right] \;\le\; \exp\Big(-\Omega\big(\delta^2 t\big)\Big).$$

*In particular, if $p \le 1/3$ and $t = O(\log(1/\varepsilon))$, then the probability that the majority of the walk samples lie in $B$ is at most $\varepsilon$.*

*Proof.* Let $X_i = \mathbf{1}_{\{v_i \in B\}}$ be the indicator that step $i$ of the walk lands in the bad set $B$. By Markov's inequality,

$$\forall \theta > 0,\ \Pr\left[\sum_{i=0}^{t-1} X_i \;\ge\; t(p + \lambda + \delta)\right] \;\le\; \frac{\mathbb{E}\Big[\exp\big(\theta \sum_{i=0}^{t-1} X_i\big)\Big]}{\exp\big(\theta t (p + \lambda + \delta)\big)}.$$

Now we rewrite the expectation as a quadratic form of graph matrices. Let $P$ be the random-walk matrix of $G$. Define the diagonal matrix $D_B$ with $(D_B)_{vv} = e^{\theta \cdot \mathbf{1}_{\{v \in B\}}}$. Then

$$\mathbb{E}\left[\exp\big(\theta \sum_{i=0}^{t-1} X_i\big)\right] = \frac{1}{n}\left\langle \vec{1}, (D_B P)^t \vec{1} \right\rangle \le \|D_B P\|_{op}^t\,,$$

where $\|M\|_{op} = \max_{\|x\|_2 = 1} \|Mx\|_2$.

Next we use spectral decomposition to understand the operator norm of $D_B P$. First we decompose $P = (1-\lambda)\cdot \frac{1}{n}\vec{1}\vec{1}^\top + \lambda \cdot P'$ where $\|P'\|_{op} \leq 1$. So for any $\theta \in (0,1)$,

$$\begin{aligned}
\|D_B P\|_{op} &\leq (1-\lambda)\cdot \left\| \frac{1}{n} D_B \vec{1}\vec{1}^\top \right\|_{op} + \lambda \cdot \|D_B P'\|_{op} \\
&\leq (1-\lambda)\cdot \sqrt{\frac{1}{n}\cdot \sum_{v\in V} e^{2\theta\cdot \mathbf{1}_{\{v\in B\}}}} + \lambda \cdot \|D_B\|_{op}\cdot \|P'\|_{op} \\
&\leq (1-\lambda)\sqrt{pe^{2\theta} + (1-p)} + \lambda e^{\theta} \\
&\leq (1-\lambda) e^{p\theta + O(\theta^2)} + \lambda e^{\theta} \\
&\leq e^{(1-\lambda)(p\theta + O(\theta^2)) + \lambda\theta + O(\theta^2)} \leq e^{\theta(p+\lambda) + O(\theta^2)}
\end{aligned}$$

The last two inequalities are due to the following approximation for the linear combinations of the exponential functions: for $a, b, c \in (0,1)$

$$\begin{aligned}
c\, e^a + (1-c)\, e^b &= c\,(1 + a + \frac{a^2}{2!} + \frac{a^3}{3!}\dots) + (1-c)\,(1 + b + \frac{b^2}{2!} + \frac{b^3}{3!}\dots) \\
&\leq c\,(1 + a + O(a^2)) + (1-c)\,(1 + b + O(b^2)\dots) \\
&= 1 + (c\,a + (1-c)\,b) + O(a^2 + b^2) \\
&\leq e^{c\,a + (1-c)\,b + O(a^2+b^2)}.
\end{aligned}$$

By this bound on the operator norm

$$\|D_B P\|_{op}^t \leq e^{\theta t(p+\lambda) + O(\theta^2 t)}.$$

Now if we set $\theta = \frac{\delta}{C}$ for some large enough constant $C > 0$, we can bound the ratio

$$\frac{\mathbb{E}\left[\exp\left(\theta \sum_{i=0}^{t-1} X_i\right)\right]}{\exp\left(\theta t(p+\lambda+\delta)\right)} \leq \exp\left(-\Omega(\delta^2 t)\right).$$

This completes the proof. $\square$

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Lecture 5: Constructions of Expanders

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 5.1 How small can $\lambda_2$ be?

Before turning to constructions of expanders, it is natural to ask: what is the smallest possible expansion parameter $\lambda$ that can be achieved by an infinite family of graphs? Since most applications involve bounded-degree graphs, we focus first on the case of $d$-regular graphs for a fixed degree $d$. This question was resolved by Alon and Boppana.

**Theorem 5.1** (Alon–Boppana Bound)**.** *For every $d$-regular graph $G$ on $n$ vertices, the second largest eigenvalue of the* ***adjacency matrix*** *satisfies*

$$\lambda_2 \geq 2\sqrt{d-1} - o_n(1).$$

*So for fixed $d$ and any $\varepsilon > 0$, there is no infinite family of $d$-regular* ***one-sided*** $\left(\frac{2\sqrt{d-1}}{d} - \varepsilon\right)$*–expanders.*

In the homework you will be asked to prove this statement. Here, we will instead establish a slightly weaker version using the celebrated trace method.

**Theorem 5.2.** *For every $d$-regular graph on $n$ vertices, the eigenvalues of the* ***adjacency matrix*** *satisfy*

$$\max(\lambda_2, |\lambda_n|) \geq 2\sqrt{d-1} - o_n(1).$$

*So for fixed $d$ and any $\varepsilon > 0$, there is no infinite family of $d$-regular* ***two-sided*** $\left(\frac{2\sqrt{d-1}}{d} - \varepsilon\right)$*–expanders.*

*Proof.* Let the adjacency matrix of $G$ be $A$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be its eigenvalues, with $\lambda_1 = d$. Define $|\lambda|_2 = \max(\lambda_2, |\lambda_n|)$.

Our goal is to find a vector $f \in \mathbb{R}^n$ such that $\langle f, A^{2k} f \rangle$ is large and $f \perp \vec{1}$. In particular, if we can show that

$$(2\sqrt{d-1} - o_n(1))^{2k} \cdot \|f\|^2 \leq \langle f, A^{2k} f \rangle,$$

then we can derive the bound

$$\begin{aligned}
(2\sqrt{d-1} - o_n(1))^{2k} \cdot \|f\|^2 &\leq \langle f, A^{2k} f \rangle \\
(2\sqrt{d-1} - o_n(1))^{2k} \cdot \|f\|^2 &= \sum_{i=2}^{n} \lambda_i^{2k} \langle f, v_i \rangle \\
(2\sqrt{d-1} - o_n(1))^{2k} \cdot \|f\|^2 &\leq |\lambda|_2^{2k} \|f\|^2 \\
|\lambda|_2 &\geq 2\sqrt{d-1} - o_n(1).
\end{aligned}$$

In the next step we construct such a function $f$. For any two distinct vertices $a, b \in G$, we can define $f_{ab} = \vec{1}_a - \vec{1}_b$. Note that $f_{ab} \perp \vec{1}$, and

$$\langle f_{ab}, A^{2k} f_{ab} \rangle = A^{2k}[a,a] + A^{2k}[b,b] - 2A^{2k}[a,b].$$

Note that $A^{2k}[a,a]$ counts the number of closed walks of length $2k$ that start and end at vertex $a$ in $G$. This quantity is at least the number of closed walks of length $2k$ in the infinite $d$-regular tree $T_d$. Thus, to obtain a lower bound, it suffices to count the number of closed walks of length $2k$ that start and end at the root of $T_d$.

Each such walk can be uniquely encoded by a pair of strings $(x, y)$ where

$$x \in \{\pm\}^{2k}, \qquad y \in [d]^k.$$

The $i$-th symbol of $x$ indicates whether the $i$-th step of the walk moves away from the root $(+)$ or toward the root $(-)$. The $j$-th symbol of $y$ records, on the $j$-th occasion that the walk moves away from the root, which of the $d$ (or $d-1$) child nodes is chosen. Observe that any string in $[d-1]^k$ can be $y$, while $x$ must be a string whose prefix sums are nonnegative (so the walk never moves closer to the root than the root itself).

By standard counting arguments, there are $\geq (d-1)^k$ possible choices for $y$, and $\frac{1}{k+1}\binom{2k}{k}$ possible choices for $x$. Therefore

$$\langle f_{ab}, A^{2k} f_{ab} \rangle \geq 2(d-1)^k \cdot \frac{1}{k+1}\binom{2k}{k} - 2A^{2k}[a,b].$$

The bound becomes tighter if $k$ is larger so we pick $a, b$ to be the vertices in $G$ that are farthest apart from each other. In this case we set $k = \lfloor (\mathrm{Dia}(G) - 1)/2 \rfloor \geq \lfloor (\log_d n - 1)/2 \rfloor$, where $\mathrm{Dia}(G)$ is the diameter of $G$. As a result

$$\begin{aligned}
\langle f_{ab}, A^{2k} f_{ab} \rangle &\geq 2(d-1)^k \cdot \frac{1}{k+1}\binom{2k}{k} - 0 \\
&\geq 2(d-1)^k \cdot \frac{2^{2k}}{(2k+1)(k+1)} \\
&= (2\sqrt{d-1} \cdot (1 - O(\log k/k))^{2k} \, \|f_{ab}\|^2 \\
&= (2\sqrt{d-1} - O(\log\log n / \log n))^{2k} \, \|f_{ab}\|^2
\end{aligned}$$

Therefore, $|\lambda|_2 \geq 2\sqrt{d-1} - o_n(1)$. □

How tight is the lower bound from the Alon–Boppana theorem? Are there graphs that achieve it? If so, can they be constructed explicitly? In fact, such graphs exist and are defined as follows.

**Definition 5.3** (Ramanujan Graphs)**.** *A $d$-regular graph $G$ is called* Ramanujan *if the second largest eigenvalue of its adjacency matrix in absolute value satisfies*

$$|\lambda|_2 \;\leq\; 2\sqrt{d-1}.$$

Lubotzky–Phillips–Sarnak and Margulis independently gave explicit constructions of Ramanujan graphs when $d-1$ is prime, and Morgenstern later extended these constructions to the case where $d-1$ is a prime power. For general degrees $d$, however, no explicit construction of Ramanujan graphs is currently known.

If one considers the bipartite version, where the condition is only that $\lambda_2 \leq 2\sqrt{d-1}$, Marcus–Spielman–Srivastava proved that bipartite Ramanujan graphs exist for every $d \geq 3$. Afterwards, Cohen provided an efficient construction algorithm for these graphs.

If we relax the definition slightly, more is known.

**Definition 5.4** (Near Ramanujan Graphs)**.** *A family of $d$-regular graphs $\{G_n\}$ is $\varepsilon$–*near–Ramanujan *if for every there exists $n_\varepsilon$ such that for all $n \geq n_\varepsilon$, the second largest eigenvalue of $G_n$ in absolute value satisfies*

$$|\lambda|_2 \ \leq \ 2\sqrt{d-1} + \varepsilon.$$

Friedman proved that for every $d \geq 3$ and $\varepsilon > 0$, random $d$-regular graphs are $\varepsilon$-near–Ramanujan with high probability.

Moreover for every $d \geq 3$ and $\varepsilon$, Mohanty–O'Donnell–Paredes gave an efficient construction of families of $d$-regular $\varepsilon$-near–Ramanujan graphs.

**Remark 5.5.** *A general version of the Alon–Boppana bound for broader classes of graphs was given by Greenberg and Lubotzky. Writing notes on this result could make for a good final project.*

# Lecture 6: An Explicit Construction of Expanders

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 6.1 Random constructions

Though the proof that random $d$-regular graphs are Ramanujan with constant probability is quite advanced, one can still use elementary techniques to show that random to prove that random $d$-regular graphs are one-sided $\frac{31}{32}$-expanders with high probability. This follows from a lower bound on the conductance of random $d$-regular graphs, obtained via a union bound argument. We state the theorem and leave its proof as an exercise.

**Theorem 6.1.** *For every fixed $d \geq 3$, a uniformly random $d$-regular graph on $n$ vertices is a one-sided $\frac{31}{32}$-expander with probability $1 - o(1)$.*

## 6.2 Magulis–Gabber–Galil expanders

We now move to the first explicit construction of expanders due to Margulis that was later analyzed by Gabber and Galil.

**Definition 6.2** (Margulis–Gabber–Galil Graph)**.** *Let $n \in \mathbb{N}$. The Margulis–Gabber–Galil graph $G_n = (V, E)$ is defined as follows:*

$$V = \mathbb{Z}_n \times \mathbb{Z}_n,$$

*and each vertex $(x, y) \in V$ is connected to the following 8 neighbors (all coordinates modulo $n$):*

$$(x, y) \mapsto (x \pm y,\, y), (x, y) \mapsto (x \pm 1,\, y), (x, y) \mapsto (x,\, y \pm x), (x, y) \mapsto (x,\, y \pm 1).$$

*Thus $G_n$ is an 8-regular graph on $n^2$ vertices.*

In this section we prove that $G_n$ is an expander:

**Theorem 6.3** (Gabber–Galil)**.** *The Margulis–Gabber–Galil graph has spectral gap (second smallest eigenvalue of the normalized Laplacian) $\gamma(G_n) \geq \frac{1}{600}$.*

To present the proof we need to define two more graphs $R_n$ and $Z$.

**Definition 6.4.** *Let $R_n$ be the uncountable graph with vertex set $V = [0, n)^2$ and edges of the form $(x, y) \to (x \pm y,\, y), (x, y) \to (x,\, y \pm x)$.*

Though this is a graph with uncountable number of vertices, we can still define functions over the graph, Laplacian operators, inner product, and spectral gap.

For any function $f, g : V \to \mathbb{R}$, define the inner product be

$$\langle f, g \rangle = \int_{[0,n)^2} f(x,y) \cdot g(x,y)\, dx\, dy$$

Laplacian operator be

$$\left\langle f, \bar{L}_R\, g \right\rangle = \frac{1}{4} \int_{[0,n)^2} \left(f(x,y) - g(x, x+y)\right)^2 + \left(f(x,y) - g(x+y, y)\right)^2\, dx\, dy,$$

and the spectral gap be

$$\gamma(R_n) = \inf_{f \,|\, \langle f, \vec{1} \rangle = 0} \frac{\left\langle f, \bar{L}_R\, f \right\rangle}{\langle f, f \rangle}.$$

The next graph we will consider is also an infinite graph.

**Definition 6.5.** *Let $Z$ be the uncountable graph with vertex set $V = \mathbb{Z}^2 \setminus \{(0,0)\}$ and edges of the form $(x,y) \to (x \pm y,\, y), (x,y) \to (x,\, y \pm x)$.*

Unlike $R_n$, $Z$ is a discrete graph we can define normalized Laplacian operator just as before. The only difference is that instead of considering all function over $V$, we only focus on functions with bounded norm. As a result the spectral gap is defined as follows

$$\gamma(Z) = \inf_{f \,|\, \|f\|_2 < \infty} \frac{\left\langle f, \bar{L}_Z f \right\rangle}{\langle f, f \rangle}.$$

Now to prove the theorem it boils down to connect the spectral gaps of the three graphs using the following lemmas.

**Lemma 6.6.**

$$\gamma(G_n) \geq \frac{1}{3} \gamma(R_n).$$

This first lemma we will skip the proof and leave it as an exercise.

**Lemma 6.7.**

$$\gamma(R_n) \geq \gamma(Z).$$

**Lemma 6.8.**

$$\phi(Z) \geq \frac{1}{10}.$$

Now we move on to prove these two lemmas. To prove Lemma 6.7 we recall some basic results from Fourier analysis.

### 6.2.1 Fourier analysis of bivariate polynomials

Let $n \in \mathbb{N}$ and consider functions $f : [0,n)^2 \to \mathbb{R}$. The Fourier characters on $\mathbb{Z}^2$ are

$$\chi_{k,\ell}(x,y) = \frac{1}{n} \exp(2\pi i(kx + \ell y))\,, \qquad (k,\ell) \in \mathbb{Z}^2.$$

They form an orthonormal basis with respect to the inner product

$$\langle f, g \rangle = \int_{[0,n)^2} f(x,y) \cdot g(x,y)\, dx\, dy$$

The Fourier coefficients of $f$ are defined as

$$\widehat{f}(k,\ell) = \int_{[0,n)^2} f(x,y) \cdot \chi_{k,\ell}(x,y)\, dx\, dy.$$

The Fourier expansion of $f$ is then

$$f(x,y) = \sum_{k,\ell \in \mathbb{Z}^2}^{n-1} \widehat{f}(k,\ell)\, \chi_{k,\ell}(x,y).$$

Then the Parseval identity gives that for any $f : [0,n)^2 \to \mathbb{R}$,

$$\langle f, f \rangle = \sum_{k,\ell \in \mathbb{Z}^2} \widehat{f}(k,\ell)^2.$$

### 6.2.2 Proof of Lemma 6.7

The main idea is to show that for any $f : [0,n)^2 \to \mathbb{R}$ with bounded norm, let $\widehat{f} : \mathbb{Z}^2 \to \mathbb{R}$ be its Fourier coefficient function, then the two functions satisfy

$$\frac{\langle f, \bar{L}_R\, f \rangle}{\langle f, f \rangle} = \frac{\left\langle \widehat{f}, \bar{L}_Z \widehat{f} \right\rangle}{\left\langle \widehat{f}, \widehat{f} \right\rangle}.$$

From the Parseval identity its clear that the two denominators are equal, we now establish the equality between the two numerators.

Define $s(x,y) = f(x,y) - f(x, x+y)$ and $t(x,y) = f(x,y) - f(x+y, y)$, then

$$\begin{aligned}
\langle f, \bar{L}_R\, f \rangle &= \frac{1}{4} \int_{[0,n)^2} (f(x,y) - f(x,x+y))^2 + (f(x,y) - f(x+y,y))^2\, dx\, dy \\
&= \frac{1}{4} \int_{[0,n)^2} s(x,y)^2 + t(x,y)^2\, dx\, dy \\
&= \frac{1}{4} \sum_{k,\ell \in \mathbb{Z}^2} \widehat{s}(k,\ell)^2 + \widehat{t}(k,\ell)^2.
\end{aligned}$$

Furthermore, by Fourier transform

$$\begin{aligned}
\widehat{s}(k,\ell) &= \int_{[0,n)^2} (f(x,y) - f(x,x+y))\, \chi_{k,\ell}(x,y)\, dx\, dy \\
&= \widehat{f}(k,\ell) - \frac{1}{n^2} \int_{[0,n)^2} f(x,x+y) \cdot \exp(2\pi i(kx + \ell y))\, dx\, dy \\
&= \widehat{f}(k,\ell) - \frac{1}{n^2} \int_{[0,n)^2} f(x,x+y) \cdot \exp(2\pi i((k-\ell)x + \ell(x+y)))\, dx\, dy \\
&= \widehat{f}(k,\ell) - \widehat{f}(k-\ell, \ell).
\end{aligned}$$

Similarly we can dedue that $\widehat{t}(k,\ell) = \widehat{f}(k,\ell) - \widehat{f}(k,\ell-k)$. Plugging in the two values, we get that

$$\begin{aligned}\langle f, \bar{L}_R f\rangle &= \frac{1}{4}\sum_{k,\ell\in\mathbb{Z}^2}\left(\widehat{f}(k,\ell)-\widehat{f}(k-\ell,\ell)\right)^2+\left(\widehat{f}(k,\ell)-\widehat{f}(k,\ell-k)\right)^2 \\ &= \left\langle \widehat{f}, \bar{L}_Z \widehat{f}\right\rangle\end{aligned}$$

Thus we complete the proof.

### 6.2.3 Proof of Lemma 6.8

For any subset $A \subseteq \mathbb{Z}^2 \setminus \{(0,0)\}$, we partition $A$ into 5 parts, $A_0 \sqcup A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4$ where $A_0 = A \cap (x\text{-axis} \cup y\text{-axis})$, and $A_i = A \cap i\text{-th quadrant}$ for $i \geq 1$.

For every set $A_i$ define

$$\begin{aligned} S(A_i) &= \{(x+y,y) \mid (x,y)\in A_i\}, \qquad & S^{-1}(A_i) &= \{(x-y,y)\mid (x,y)\in A_i\} \\ T(A_i) &= \{(x,y+x)\mid (x,y)\in A_i\}, \qquad & T^{-1}(A_i) &= \{(x,y-x)\mid (x,y)\in A_i\}.\end{aligned}$$

We observe that (1)

$$|S(A_i)| = \left|S^{-1}(A_i)\right| = |A_i| = |T(A_i)| = \left|T^{-1}(A_i)\right|,$$

and (2) $S(A_1), T(A_1)$ are in the 1-st quadrant, $S^{-1}(A_2), T^{-1}(A_2)$ are in the 2-nd quadrant, $S(A_3), T(A_3)$ are in the 3-rd quadrant, and $S^{-1}(A_4), T^{-1}(A_4)$ are in the 4-th quadrant.

Furthermore these 8 sets are disjoint since

$$S(A_1)\cap T(A_1) \neq \emptyset \Rightarrow \exists (x,y),(a,b)\in A_i \text{ s.t. } \begin{cases} x+y=a \\ y=a+b\end{cases} \Rightarrow x=-b \Rightarrow (x,y),(a,b) \text{ cannot be both in } A_1.$$

Similar arguments apply to the other $i$'s. As a result we have

$$\left|E(A\setminus A_0, \bar{A})\right| \geq |A\setminus A_0| = |A| - |A_0|.$$

For $(x,y)\in A_0$, 2 of edges are self loops while the other 2 leave the axes. As a result

$$\left|E(A_0,\bar{A})\right| = \left|E(A_0,\bar{A}_0)\right| - |E(A_0, A\setminus A_0)| \geq 2|A_0| - 2|A\setminus A_0| = 4|A_0| - 2|A|.$$

Combine the two lower bounds to get

$$\left|E(A,\bar{A})\right| \geq \frac{2}{5}|A| \Rightarrow \phi(Z) \geq \frac{1}{10}.$$

# Lecture 7: An Iterative Construction of Bipartite Expanders

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 7.1 2-lift of a graph

**Definition 7.1** (2-Lift of a Graph)**.** *Let $G = (V, E)$ be a graph. A* 2-lift *of $G$ is a graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ defined as follows:*

- *The vertex set is a double copy of $V$:*

$$\widetilde{V} = V \times \{+1, -1\}.$$

- *For each edge $\{u, v\} \in E$, we choose a* sign *$\sigma(u, v) \in \{+1, -1\}$.*
  - *If $\sigma(u, v) = +1$, then we add parallel edges $(u, +1) \sim (v, +1)$, $(u, -1) \sim (v, -1)$ to $\widetilde{E}$.*
  - *If $\sigma(u, v) = -1$, then we add cross edges $(u, +1) \sim (v, -1)$, $(u, -1) \sim (v, +1)$ to $\widetilde{E}$.*

*Thus the choice of signs $\{\sigma(u, v)\}_{(u,v)\in E}$ uniquely determines the 2-lift $\widetilde{G}$. Furthermore, $\widetilde{G}$ has twice the number of vertices of $G$ and the same average degree as $G$.*

The spectrum of $G$ and that of $\widetilde{G}$ are closed related as evident from the following lemma.

**Lemma 7.2.** *Let $G = (V, E)$ be a graph with adjacency matrix $A \in \{0, 1\}^{n\times n}$. Fix a signing $\sigma : E \to \{\pm 1\}$ and let $A^\sigma$ denote the* signed adjacency matrix*, i.e.*

$$A^\sigma_{uv} = \begin{cases} \sigma(u, v) & \text{if } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

*Let $\widetilde{G}$ be the 2-lift of $G$ corresponding to the signing $\sigma$, with adjacency matrix $\widetilde{A}$. Then the multiset of eigenvalues of $\widetilde{A}$ is the disjoint union of the eigenvalues of $A$ and the eigenvalues of $A^\sigma$.*

*Proof.* Order the vertices of the 2-lift as $\{(v, +1) : v \in V\}$ followed by $\{(v, -1) : v \in V\}$. With this ordering, the adjacency matrix of the lift can we written as

$$\widetilde{A} = \frac{1}{2} \cdot \begin{bmatrix} A + A^\sigma & A - A^\sigma \\ A - A^\sigma & A + A^\sigma \end{bmatrix}.$$

Now observe that $\mathbb{R}^{2n}$ is the direct sum of the *symmetric* and *antisymmetric* subspaces:

$$U_+ = \{(x, x) : x \in \mathbb{R}^n\}, \qquad U_- = \{(x, -x) : x \in \mathbb{R}^n\}.$$

*Case 1 (symmetric subspace).* Take a vector $(x, x) \in U_+$. Then

$$\widetilde{A} \begin{bmatrix} x \\ x \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} A + A^\sigma & A - A^\sigma \\ A - A^\sigma & A + A^\sigma \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} Ax \\ Ax \end{bmatrix}.$$

Thus $U_+$ is invariant under $\widetilde{A}$ and the restriction of $\widetilde{A}$ to $U_+$ is exactly $A$.

*Case 2 (antisymmetric subspace).* Take a vector $(x, -x) \in U_-$. Then

$$\widetilde{A}\begin{bmatrix} x \\ -x \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} A + A^\sigma & A - A^\sigma \\ A - A^\sigma & A + A^\sigma \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} A^\sigma x \\ -A^\sigma x \end{bmatrix}.$$

Thus $U_-$ is invariant under $\widetilde{A}$ and the restriction of $\widetilde{A}$ to $U_-$ is exactly $A^\sigma$.

Since $A$ is the adjacency matrix of $G$ and $A^\sigma$ is the signed adjacency matrix, the eigenvalues of $\widetilde{A}$ are precisely the union of the eigenvalues of $A$ and of $A^\sigma$. □

## 7.2 An iterative construction of expanders

Bilu and Linial suggested the following way of constructing expanders via 2-lifts.

1. Start with a constant size $d$-regular expander $G$.
2. Randomly sign its edges, producing a signed adjacency matrix $A^\sigma$.
3. Iterating this process gives an infinite family of bounded-degree graphs.

They proved the following fact on the spectrum of a random 2-lift.

**Theorem 7.3.** *Let $G$ be a $d$-regular graph with adjacency matrix $A$. There exists a signing $\sigma : E(G) \to \{\pm 1\}$ such that the signed adjacency matrix $A^\sigma$ satisfies*

$$\|A^\sigma\|_{op} \ = \ O\left(\sqrt{d \log^3 d}\right).$$

*So if $G$ is a $d$-regular expander, its 2-lift $\widetilde{G}$ corresponding to $\sigma$ is also a $d$-regular expander.*

This existential result can be made algorithmic, and as a result Bilu and Linial gave an efficient algorithm that constructs an infinite family of $d$-regular expanders.

## 7.3 Existence of bipartite Ramanujan graphs

Inspired by this construction, Marcus–Spielman–Srivastava proved existence of an infinite family of **bipartite $d$-regular Ramanujan graphs**.

One can start with a constant size bipartite $d$-regular Ramanujan graph. Then it boils down to prove that there exists a signing $\sigma$ such that the eigenvalues of $A^\sigma$ are in $[-d, 2\sqrt{d-1}]$.

**Theorem 7.4.** *For every $d$-regular graph $G$, there exists a signing $\sigma$ such that the eigenvalues of the signed adjacency matrix $A^\sigma$ are all contained in the interval*

$$[-d,\, 2\sqrt{d-1}].$$

*Hence, if $G$ is a bipartite $d$-regular Ramanujan graph, then the corresponding 2-lift is also a bipartite $d$-regular Ramanujan graph.*

To prove the theorem, first recall that the eigenvalues of $A^\sigma$ are the roots of the characteristic polynomial $\det(xI - A^\sigma)$. A classical result due to Godsil stated that the expectation of the characteristic polynomial over random signing is the matching polynomial of $G$.

**Lemma 7.5** (Godsil)**.** *Let $G = (V, E)$ be a graph on $n$ vertices. Then*

$$\mathbb{E}_\sigma[\det(xI - A^\sigma)] = \mu_G(x),$$

*where*

$$\mu_G(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k \, x^{n-2k},$$

*and $m_k$ is the number of $k$-matchings in $G$.*

*Proof.* Expand the determinant as:

$$\det(xI - A^\sigma) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} (xI - A^\sigma)_{i,\pi(i)}.$$

Now take the expectation over random signing. Any monomial in the expansion is a product of entries of $A^\sigma$ and a power of $x$. Because the signs are independent and have mean zero, the expectation vanishes unless every edge appears an even number of times.

The only permutations $\pi$ that survive this averaging are those consisting of disjoint transpositions and fixed points:

$$\pi = (i_1 \, j_1)(i_2 \, j_2) \cdots (i_k \, j_k),$$

corresponding to a $k$-matching in $G$. In such a $\pi$, each fixed point contributes a factor of $x$, while each transposition $(i, j)$ contributes a factor of $-1$ (to $\operatorname{sgn}(\pi)$).

Thus, a matching of size $k$ contributes $(-1)^k x^{n-2k}$, and the number of such terms is exactly the number of $k$-matchings in $G$.

So, summing over all $k$ eventually gives

$$\mathbb{E}_\sigma[\det(xI - A^\sigma)] = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k \, x^{n-2k} = \mu_G(x).$$

□

The Heilmann–Lieb theorem on matching polynomials says that for every $d$-regular graph $G$, the roots of $\mu_G(x)$ are in $[-2\sqrt{d-1}, \, 2\sqrt{d-1}]$.

Marcus–Spielman–Srivasta then used the method of interlacing polynomials to prove that there exists a signing $\sigma$ such that the characteristic polynomial $\det(xI - A^\sigma)$ has its maximum root bounded by that of the average polynomial $\mu_G(x)$. To explain it we start with defining interlacing polynomials.

**Definition 7.6** (Interlacing polynomial)**.** *We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ interlaces a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if*

$$\beta_n \le \alpha_{n-1} \le \beta_{n-1} \le \alpha_{n-2} \le \cdots \le \alpha_1 \le \beta_1.$$

*Furthermore, we say that polynomials $f_1, \ldots, f_k$ have a common interlacing if there is a polynomial $g$ such that $g$ interlaces $f_i$ for every $i \in [k]$.*

**Lemma 7.7.** *Let $f_1, \ldots, f_k$ be polynomials of the same degree that are real-rooted and have positive leading coefficients. Define*

$$f = \sum_{i=1}^{k} f_i$$

*If $f_1, \ldots, f_k$ have a common interlacing, then there exists an $i \in [k]$ so that the largest root of $f_i$ is at most the largest root of $f$.*

*Proof.* Let $n$ be the degree of $f_i$ and $g$ be the common interlacing with roots $\alpha_{n-1} \leq \cdots \leq \alpha_1$. Since each $f_i$ has a positive leading coefficient, $f_i(x) \geq 0$ for $x \geq \beta_1^{(i)}$ which is the top root of $f_i$. Furthermore, each $f_i$ has only one root that is $\geq \alpha_1$, so $f_i(\alpha_1) \leq 0$. Subsequently $f(\alpha_1) \leq 0$. This tells us that $f$ has a root that is at least $\alpha_1$. Let $\beta_1$ be the largest root of $f$. Since $f$ is the sum of the $f_i$, there must be some $i$ for which $f_i(\beta_1) \geq 0$. As $f_i$ only has one root $\beta_1^{(i)} \geq \alpha_1$, then $\alpha_1 \leq \beta_1^{(i)} \leq \beta_1$. □

So now it suffices to prove that the characteristic polynomials have a common interlacing. We do so recursively, by first noting that

$$f_+(x) + f_-(x) = 2^{|E|} \cdot \mu_G(x)$$

where

$$f_+(x) = \sum_{\sigma:\sigma(1)=+} p_\sigma(x), \qquad f_-(x) = \sum_{\sigma:\sigma(1)=-} p_\sigma(x).$$

Then we are going to show that $f_+$ and $f_-$ have a common interlacing. Thus one of the function has maximum root bounded by $2\sqrt{d-1}$. Then recursively decompose the function with bounded maximum root until ending with a $p_\sigma$ with bounded maximum root.

To prove that the common interlacing exists, we need the following two results. Their proofs will be skipped.

**Lemma 7.8.** *Let $f, g$ be univariate polynomials of the same degree with positive leading coefficients. Then $f, g$ have a common interlacing if and only if for all convex combinations $h_t(x) = tf(x) + (1-t)g(x)$ where $t \in [0,1]$ is real-rooted.*

**Lemma 7.9.** *Consider a graph $G$ with $m$ edges. For any $c_1, \ldots, c_m \in [0,1]^m$, the following polynomial is real-rooted*

$$\sum_{\sigma \in \{\pm\}^m} \prod_{i:\sigma(i)=+} c_i \cdot \prod_{i:\sigma(i)=-} (1-c_i) \cdot p_\sigma(x).$$

Finally as a corollary we get that:

**Corollary 7.10** (Restatement of Theorem 7.4)**.** *Let $G = (V,E)$ be a $d$-regular graph. For each signing $\sigma : E \to \{\pm 1\}$, let $p_\sigma(x)$ be the characteristic polynomial of the signed adjacency matrix $A^\sigma$. Then there exists a signing $\sigma$ such that the largest root of $p_\sigma(x)$ is at most $2\sqrt{d-1}$.*

**590.05 Expanders and High-dimensional Expanders** **Fall 2025**

# Lecture 8: Expanders from Zig-zag Products

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 8.1 Zig–zag product of graphs

**Definition 8.1** (Zig–Zag Product). *Let $G$ be a $D$-regular graph on vertex set $[N]$, with its edges labeled by $[D]$. Let $H$ be a $d$-regular graph on vertex set $[D]$.*

*The* zig–zag product $G \text{ z } H$ *is the $d^2$-regular graph on vertex set*

$$V(G \text{ z } H) \;=\; [N] \times [D].$$

*An edge is defined as follows. From a vertex $(v, i) \in [N] \times [D]$:*

1. ***Zig:*** *Move inside $H$ from $i$ to $j$, i.e. $(v, i) \to (v, j)$.*
2. ***Zag:*** *Cross the edge of $G$ incident to $v$ labeled $j$, arriving at $(u, j)$.*
3. ***Zig:*** *Move inside $H$ again, from $j$ to $k$, arriving at $(u, k)$.*

*Thus each edge of $G \text{ z } H$ is of the form*

$$(v, i) \;\longrightarrow\; (u, k),$$

*where $(i, j)$ and $(j, k)$ are edges of $H$, and $(v, u)$ is the adjacent edge of $v$ labeled $j$.*

## 8.2 Zig–zag product expanders

In this section we prove that zig–aag products preserve expansion of the two original graphs.

**Theorem 8.2** (Reingold–Vadhan–Wigderson). *Let $G$ be a $D$-regular graph on $N$ vertices with second-largest eigenvalue $\lambda_G$, and let $H$ be a $d$-regular graph on $D$ vertices with second-largest eigenvalue in absolute value $\lambda_H$. Then the zig–zag product $G \text{ z } H$ is a $d^2$-regular graph on $N \cdot D$ vertices whose second-largest eigenvalue satisfies*

$$\lambda(G \text{ z } H) \;\le\; \lambda_G + \lambda_H + \lambda_H^2.$$

*Proof.* Let $A_G$ be the normalized adjacency matrix of $G$ (an $N \times N$ matrix), and $A_H$ the normalized adjacency matrix of $H$ (a $D \times D$ matrix).

Define the permutation matrix $P$ of size $ND \times ND$ that performs the "zag" step: it maps $(v, j)$ to $(u, j)$ where $(v, u)$ is the edge of $G$ labeled $j$. The "zig" steps are implemented by $I_N \otimes A_H$, acting on $[N] \times [D]$.

Thus the normalized adjacency matrix of $G \text{ z } H$ is

$$A = (I_N \otimes A_H)\, P\, (I_N \otimes A_H).$$

Now do a spectral decomposition of $A_H$ as

$$A_H = \tfrac{1}{D} J + B,$$

where $J$ is the all-ones matrix and $\|B\| \le \lambda_H$. Substitute the decomposition:

$$A = (I \otimes (\tfrac{1}{D} J + B))\, P\, (I \otimes (\tfrac{1}{D} J + B)).$$

Expanding gives four terms:

$$A = (I \otimes \tfrac{1}{D} J)\, P\, (I \otimes \tfrac{1}{D} J) + (I \otimes B)\, P\, (I \otimes \tfrac{1}{D} J) + (I \otimes \tfrac{1}{D} J)\, P\, (I \otimes B) + (I \otimes B)\, P\, (I \otimes B).$$

For every $x \perp \vec{1}$ in $\mathbb{R}^{ND}$ we decompose $x = x_{\|} \otimes \vec{1}_D + x_{\perp}$ where

$$x_{\|}(v) = \frac{1}{D} \sum_{j=1}^{D} x(v, j)\,, \quad x_{\perp}(v, i) = x(v, i) - x_{\|}(v).$$

Notice that for all $v$

$$\sum_{i=1}^{D} x_{\perp}(v, i) = \sum_{i=1}^{D} x(v, i) - \sum_{i=1}^{D} x_{\|}(v, i) = 0.$$

Therefore $x_{\perp}(v) \perp \vec{1}_D$ and $x_{\|} \perp \vec{1}_N$. As a result

$$(I \otimes \tfrac{1}{D} J)\, x_{\perp} = \vec{0}, \quad (I \otimes B)\, x_{\|} \otimes \vec{1}_D = \vec{0}$$

$$(I \otimes \tfrac{1}{D} J)\, x_{\|} \otimes \vec{1}_D = x_{\|} \otimes \vec{1}_D, \quad (I \otimes B)\, x_{\perp} = \begin{bmatrix} B x_{\perp}(v_1) \\ \vdots \\ B x_{\perp}(v_N) \end{bmatrix}$$

Then for each of the four terms we can get that

$$\left\langle x, (I \otimes \tfrac{1}{D} J)\, P\, (I \otimes \tfrac{1}{D} J) x \right\rangle = \left\langle x_{\|} \otimes \vec{1}_D, P x_{\|} \otimes \vec{1}_D \right\rangle = D \cdot \left\langle x_{\|}, A_G x_{\|} \right\rangle \le \lambda_G \|x_{\|} \otimes \vec{1}_D\|_2^2$$

$$\begin{aligned} \left\langle x, (I \otimes B)\, P\, (I \otimes \tfrac{1}{D} J) x \right\rangle &= \left\langle x_{\perp}, (I \otimes B)\, P\, (I \otimes \tfrac{1}{D} J)\, x_{\|} \otimes \vec{1}_D \right\rangle \\ &= \begin{bmatrix} B x_{\perp}(v_1) & \dots & B x_{\perp}(v_N) \end{bmatrix} \, P\, x_{\|} \otimes \vec{1}_D \\ &\le \lambda_H \, \|x_{\perp}\|_2 \cdot \left\| x_{\|} \otimes \vec{1}_D \right\|_2 \end{aligned}$$

$$\langle x, (I \otimes B)\, P\, (I \otimes B) x \rangle = \begin{bmatrix} B x_{\perp}(v_1) & \dots & B x_{\perp}(v_N) \end{bmatrix} \, P \, \begin{bmatrix} B x_{\perp}(v_1) \\ \vdots \\ B x_{\perp}(v_N) \end{bmatrix} \le \lambda_H^2 \, \|x_{\perp}\|_2^2 \,.$$

Thus, combining all four terms together, we have that for every $x$ in the space orthogonal to $\vec{1}$, the operator norm of $A$ is bounded by

$$\frac{\langle x, Ax \rangle}{\|x\|_2^2} \le \frac{\lambda_G \left\| x_{\|} \otimes \vec{1}_D \right\|_2^2 + 2\lambda_H \left\| x_{\|} \otimes \vec{1}_D \right\|_2 \cdot \|x_{\perp}\|_2 + \lambda_H^2 \, \|x_{\perp}\|_2^2}{\left\| x_{\|} \otimes \vec{1}_D \right\|_2^2 + \|x_{\perp}\|_2^2} \le \lambda_G + \lambda_H + \lambda_H^2.$$

□

As a result we can construct infinite families of $\Delta$-regular graphs in the following way.

1. Start with a $\sqrt{\Delta}$-regular graph $H$ over $\Delta^2$ vertices.
2. Let $G_1 = H^2$ then $G_1$ has degree $\Delta$ and expansion $\lambda_H^2$.
3. Iteratively construct $G_{i+1} = G_i^2 \,\text{z}\, H$. Note that this is a valid zig–zag product since the degree of $G_i^2$ and the vertex number in $H$ are both $\Delta^2$.

Note that if we want to construct $\lambda$-expanders, then it suffices to pick $H$ to be $\lambda$-expander. Since in this case

$$\lambda(G_1) = \lambda_H^2 \le \lambda^2, \quad \lambda(G_{i+1}) \le \lambda(G_i)^2 + \lambda_H(1+\lambda_H) \le \lambda^2 + \lambda(1+\lambda) \le \lambda.$$

## 8.3 Expansion of $G(n, p)$

As a final part of expanders, we use trace method to compute the expansion of Erdös-Renyí graphs.

**Theorem 8.3.** *Let $G \sim G(n,p)$ be an Erdős–Rényi random graph on $n$ vertices. Assume $np \ge 2\log^8 n$. Then with high probability*

$$|\lambda|_2(G) \;\le\; (1+o_n(1)) \cdot 2\sqrt{np},$$

*where $|\lambda|_2(G)$ denotes the second-largest eigenvalue in absolute value of the adjacency matrix of $G$.*

*Proof.* The adjacency matrix $A$ of $G(n,p)$ can be decomposed as

$$A \;=\; \mathbb{E}[A] + R,$$

where $\mathbb{E}[A] = p(J-I)$ and $R = A - \mathbb{E}[A]$ is a centered random symmetric matrix with independent entries. We further observe the spectral connection between $A$ and $R$:

$$|\lambda|_2(G) \le \|A - pJ\|_{op} \le \|A - \mathbb{E}[A]\|_{op} = \|R\|_{op}.$$

Thus it suffices to show w.h.p.

$$\|R\|_{op} = (1+o(1))2\sqrt{np}.$$

**Trace moment method.** Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $R$. For any even integer $2k$,

$$\|R\|_{op}^{2k} \;\le\; \mathrm{Tr}(R^{2k}) = \sum_{i=1}^{n} \lambda_i^{2k}.$$

Taking expectation,

$$\mathbb{E}\,\mathrm{Tr}(R^{2k}) = \sum_{i=1}^{n} \mathbb{E}[\lambda_i^{2k}].$$

**Combinatorial interpretation.** We can expand trace moments as sum of weighted walks

$$\mathrm{Tr}(R^{2k}) = \sum_{v_0, v_1, \dots, v_{2k-1}} R_{v_0 v_1} R_{v_1 v_2} \cdots R_{v_{2k-1} v_0}.$$

Taking expectation, independence of entries implies that only terms corresponding to closed walks in which every edge is traversed at least twice (no singleton edges) survive. Each such walk uses at most $k+1$ vertices and contributes at most $p^k$ to the expectation.

**Bounding the number of closed walks.** The number of closed walks of length $2k$ without a singleton edge on exactly $s$ vertices is bounded by

$$C_s \leq n^s \binom{2k}{s-1}\binom{2k-(s-1)}{s-1} s^{2k-2(s-1)}$$

- $n^s$: choose the $s$ vertices $v_1, \dots, v_s$ in the walk.
- $\binom{2k}{s-1}$: location of steps reaching a new vertex. Call these edges the tree edges.
- $\binom{2k-(s-1)}{s-1}$: location of second time traversing the tree edges.
- $s^{2k-2(s-1)}$: all other steps and the visited vertices they are reaching.

In the case $s = k+1$

$$n^s \binom{2k}{s-1}\binom{k}{s-1} \cdot s^{2k-2(s-1)} \leq n^{k+1} 2^{2k}$$

For all $s = 2 \dots, k$

$$\frac{C_s}{C_{s+1}p} \leq \frac{1}{np} \cdot \frac{s}{(2k-s+1)} \cdot \frac{s(2k-s+1)}{(2k-2s+2)(2k-2s+1)} \cdot (s+1)^2 \leq \frac{k^2(s+1)^2}{np} \leq O\left(\frac{1}{\log^4 n}\right).$$

The last inequality follows from setting $k = \Theta(\log n)$ and $p \geq \frac{\log^8 n}{n}$. Therefore in the trace moment calculation the term with $s = k+1$ dominates,

$$\mathbb{E}\operatorname{Tr}(R^{2k}) \leq \sum_{s=2}^{k+1} C_s p^{s-1} = \left(1 + O\left(\frac{1}{\log^4 n}\right)\right) C_{k+1} p^k \leq (1+o(1)) \cdot n(4np)^k.$$

**Markov's inequality.** Set $k = c \log n$ for a small constant $c > 0$. Then by Markov's inequality,

$$\Pr[\|R\| \geq (1+\varepsilon) 2\sqrt{np}] \;\leq\; \frac{\mathbb{E}\operatorname{Tr}(R^{2k})}{((1+\varepsilon)^{2k}(4np)^k)}.$$

Plugging in the bound gives

$$\Pr[\|R\| \geq (1 + 1/o(\log n)) 2\sqrt{np}] \leq e^{-\omega(1)}.$$

Therefore, with high probability

$$\|R\| \leq (1+o(1)) \cdot 2\sqrt{np}.$$

This gives

$$\lambda_2(G) = (1+o(1)) \cdot 2\sqrt{np}.$$

□

**590.05 Expanders and High-dimensional Expanders** **Fall 2025**

# Lecture 9: Introduction to High-dimensional Expanders

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 9.1 Simplicial complexes

We generalize expansion to hypergraphs $(V, \mathcal{E})$, where $\mathcal{E}$ is a set of hyperedges.

**Definition 9.1** (Simplicial complex)**.** *A simplicial complex $X$ is a hypergraph with downward closure. I.e. for every hyperedge $\tau \in X$, the subedges $\sigma \subseteq s$ are also in $X$.*

*We denote the set hyperedges of size $(i+1)$ as $X(i) \subseteq \binom{V}{i+1}$. A hyperedge $\sigma \in X(i)$ is also called an $i$-face.*

*The dimension $d$ of $X$ is the largest $i$ such that $X(i) \neq \varnothing$.*

We will mainly consider pure simplicial complexes.

**Definition 9.2** (Pure simplicial complex)**.** *A $d$-dimensional simplicial complex $X$ is* pure *if for every $\sigma \in \bigcup_{i=0}^{d-1} X(i)$ there exists a $\tau \in X(d)$ such that $\sigma \subseteq \tau$. In other words, maximal faces induce all lower-dimensional faces.*

**Example 9.3.** *Here is an example of a pure 2-dimensional simplicial complex presented in two ways.*

$$X(0) = \{a, b, c, d, e, f\}, \quad X(1) = \{ab, ad, bd, cd, cf, df, be, de\}, \quad X(2) = \{abd,\ cdf,\ bde\}.$$

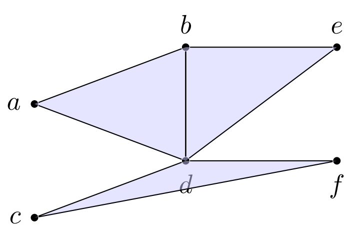


Figure 9.1: $X$

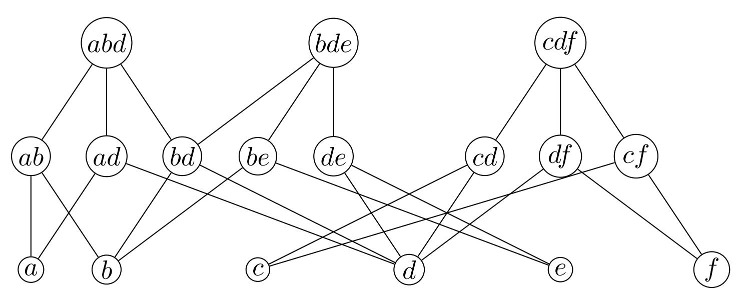


Figure 9.2: Hasse diagram

## 9.2 Links, skeletons, and their random walks

**Definition 9.4** (Link)**.** *In a $d$-dimensional simplicial complex $X$, for every $i \leq d-2$ and $\sigma \in X(i)$, the* link *of $\sigma$ is the $(d-i-1)$-dimensional simplicial complex*

$$X_\sigma = \{\, \tau \setminus \sigma \mid \tau \in X,\ \tau \supseteq \sigma \,\}.$$

**Example 9.5.** *Consider $X$ from Example 9.3:*

$$X_a = \{b, d, \{b, d\}\}, \quad X_d = \{a, b, c, e, f, \{a, b\}, \{b, e\}, \{c, f\}\}.$$

**Definition 9.6** (Skeleton)**.** *For a $d$-dimensional simplicial complex $X$ and some $k \leq d$, the $k$-skeleton of $X$ is the simplicial complex*

$$\bigcup_{i=-1}^{k} X(i),$$

*consisting of all faces of dimension at most $k$.*

### Random walks on simplicial complexes

To define random walks consistently, we assign weights to all faces of a pure simplicial complex $X$.

**Definition 9.7** (Weight function)**.** *Let $X$ be a pure $d$-dimensional simplicial complex with a weight function $w_d : X(d) \to \mathbb{R}_{\geq 0}$. For every $i = 0, \ldots, d-1$, iteratively define the induced weight on $i$-faces as*

$$w_i(\sigma) \ = \sum_{\tau \in X(i+1): \sigma \subseteq \tau} w_{i+1}(\tau), \quad \forall \sigma \in X(i).$$

Thus for $i$-face $\sigma$, its weight is the sum of the weights of $(i+1)$-faces containing it.

The link of $\sigma \in X(i)$ inherits the weight from $X$ as follows

$$\forall \tau \in X_\sigma(j), w_{\sigma,j} = w_{i+j+1}(\sigma \cup \tau).$$

Given the weight functions, we can define the random walk on the 1-skeleton of $X$ as

$$\Pr[v_1 = u \mid v_0 = v] = \frac{w_1(\{u, v\})}{w_0(v)}.$$

Analogously the walk the random walk on the 1-skeleton of $X_\sigma$ is

$$\Pr[v_1 = u \mid v_0 = v] = \frac{w_{\sigma,1}(\{u, v\})}{w_{\sigma,0}(v)} = \frac{w_{i+2}(\sigma \cup \{u, v\})}{w_{i+1}(\sigma \cup \{v\})}.$$

## 9.3 High-dimensional expanders

We want to generalize spectral expansion to simplicial complexes. In a simplicial complex, we can define expansion of 1-skeletons now that we have defined random walks on them.

**Definition 9.8** (High-dimensional expanders)**.** *A pure $d$-dimensional simplicial complex $X$ (with weight function $w_d$) is a one-sided (or two-sided) $\gamma$-expander if:*

1. *The 1-skeleton is a one-sided (or two-sided) $\gamma$-expander.*
2. *For all $i \leq d-1$ and $\sigma \in X(i)$, the 1-skeleton of $X_\sigma$ is a one-sided (or two-sided) $\gamma$-expander.*

**Remark 9.9.** *A quick check give that $d$-dim complete complexes are $d$-dim two-sided $\frac{1}{n-d-2}$-expanders while complete $d$-dim partite complexes are $d$-dim one-sided 0-expanders.*

# Lecture 10: Properties of HDXes

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 10.1 Applications

HDXes have several important applications, including:

- Matroid basis sampling
- Derandomizing direct product tests
- Locally testable codes and quantum low density parity check codes
- Improve length of PCPs.

## 10.2 Trickle-down theorem

The property that makes spectral HDXes useful is the *Trickle-Down Theorem.*

**Theorem 10.1** (Base case)**.** *Let $X$ be a 2-dimensional simplicial complex with weight $w_2$. Suppose that:*

1. *The 1-skeleton of $X$ is connected.*
2. *For every $v \in X(0)$, the 1-skeleton of the link $X_v$ is a one- or two-sided $\lambda$-expander.*

*Then the 1-skeleton of $X$ is a one- or two-sided $\frac{\lambda}{1-\lambda}$-expander.*

**Theorem 10.2** (General case, Oppenheim)**.** *Let $X$ be a $d$-dimensional simplicial complex with weight $w_d$. Suppose that:*

1. *For all $i \leq d-2$ and all $\sigma \in X(i)$, the 1-skeleton of $X_\sigma$ is connected. This includes the $(-1)$-face $\varnothing$ whose link is the 1-skeleton of $X$.*
2. *For all $\tau \in X(d-2)$, the 1-skeleton of $X_\tau$ is a one- or two-sided $\gamma$-spectral expander.*

*Then $X$ is a $d$-dimensional one- or two-sided $\frac{\lambda}{1-(d-1)\lambda}$- expander.*

*Proof (Base Case).* Let $A$ be the transition matrix of the random walk on the 1-skeleton of $X$, and let $f : X(0) \to \mathbb{R}$ be a function over the 0-faces.

We define the inner product

$$\langle f, g \rangle = \sum_{v \in X(0)} f(v) g(v)\, \pi_0(v),$$

where $\pi_0$ is the stationary distribution of $A$. Note that $\pi_0(v) = Z \cdot w_0(v)$ for some $Z \in \mathbb{R}$, since

$$A[u,v] \cdot \pi_0(v) = \frac{w_1(u,v)}{w_0(v)} \cdot Z \cdot w_0(v) = \frac{w_1(u,v)}{w_0(u)} \cdot Z \cdot w_0(u) = A[v,u]\,\pi_0(u).$$

Under this definition, the top eigenvalue of $A$ is $\lambda_1(A) = 1$, with eigenvector $\vec{1}$.

Similarly, define the link's transition matrix $A_v$ and its stationary distribution $\pi_v$. Note that $\pi_v(u) = \frac{w(u,v)}{w(v)}$.

By condition (2), for any $g : X_v(0) \to \mathbb{R}$ with $\langle g, \mathbf{1} \rangle = 0$, we have

$$\langle A_v g, g \rangle \le \lambda \|g\|^2 \quad \text{(one-sided)},$$

and

$$|\langle A_v g, g \rangle| \le \lambda \|g\|^2 \quad \text{(two-sided)}.$$

Now consider a function $f : X(0) \to \mathbb{R}$ such that

$$Af = \lambda_2 f,$$

i.e. $f$ is an eigenfunction of the second largest eigenvalue (or second largest eigenvalue in absolute value) of $A$.

Consider the restriction of $f$ to $X_u$:

$$f|_u : X_u(0) \to \mathbb{R}, \quad f|_u(v) = f(v) \quad \forall v \in X_u(0).$$

We now have the following decomposition:

$$\begin{aligned}\langle f, Af \rangle &= \mathbb{E}_{v\sim\pi}\left[f(v)(Af)(v)\right] \\ &= \mathbb{E}_{u\sim\pi}\, \mathbb{E}_{v\sim\pi_u}\left[f|_u(v)\,(A_u f|_u)(v)\right].\end{aligned}$$

Now decompose

$$f|_u = (Af)(u)\vec{1} + g_u,$$

where $(Af)(u) = \mathbb{E}_{v\sim\pi_u}[f|_u(v)] = \left\langle f|_u, \vec{1} \right\rangle_u$ and $g_u \perp \vec{1}$.

Therefore,

$$\langle f|_u, A_u f|_u \rangle = \langle g_u, A_u g_u \rangle + (Af(u))^2.$$

Plugging in this decomposition, we have

$$\langle f, Af \rangle = \mathbb{E}_{v\sim\pi}\left[\langle g_v, A_v g_v \rangle + (Af)(v)^2\right]$$

.

In the one-sided case we can deduce

$$\begin{aligned}\lambda_2\, \mathbb{E}[f^2] &\le \mathbb{E}_{v\sim\pi}\left[\lambda \|g_v\|^2\right] + \lambda_2^2\, \mathbb{E}[f^2] \\ \lambda_2 &\le \lambda(1-\lambda_2^2) + \lambda_2^2 \quad \implies \quad \lambda_2 \le \frac{\lambda}{1-\lambda}.\end{aligned}$$

In the two-sided case we can deduce

$$\lambda_2 \, \mathbb{E}[f^2] \le \left|\mathbb{E}_{v\sim\pi}\left[\lambda \|g_v\|^2\right]\right| + \left|\lambda_2^2 \, \mathbb{E}[f^2]\right|$$

$$\lambda_2 \le \lambda(1-\lambda_2^2) + \lambda_2^2 \quad \Longrightarrow \quad \lambda_2 \le \frac{\lambda}{1-\lambda}.$$

□

In fact this theorem tells us that to prove a $d$-dimensional expander is a $\lambda$-expander, it suffices to prove that the $d-2$ faces are $\frac{\lambda}{1+(d-1)\lambda}$-expanders and the lower dimensional faces' links are connected. From a different perspective, this theorem also says that for good HDXes, the global expansion (of 1-skeleton) can be witnessed by local expansion (of links' 1-skeletons).

## 10.3 Other random walks

### More operators on simplicial complexes

**Definition 10.3.** *In a $d$-dimensional complex $X$, for $i \le d$, define the $i$-chains*

$$C^i(X) = \{f : X(i) \to \mathbb{R}\}.$$

**Definition 10.4.** *For all $0 \le i \le d$, define*

$$D_i : C^i(X) \to C^{i-1}(X), \qquad (D_i f)(\sigma) = \mathbb{E}_{\tau\supset\sigma}[f(\tau)],$$

*where $\sigma \in X(i-1)$. $D_i$ can be viewed as the random walk operators on the induced bipartite graph over $X(i-1) \cup X(i)$ in the Hasse diagram.*

*Similarly, define*

$$U_{i-1} : C^{i-1}(X) \to C^i(X), \qquad (U_{i-1} g)(\tau) = \frac{1}{(i+1)} \sum_{\sigma\subset\tau} g(\sigma),$$

*where $\tau \in X(i)$. $U_{i-1}$ can be viewed as another random walk operators on the induced bipartite graph over $X(i-1) \cup X(i)$ in the Hasse diagram.*

**Remark 10.5.** *The operators can be viewed as matrices $U_i \in \mathbb{R}^{|X(i)|\times|X(i-1)|}$ and $D_i \in \mathbb{R}^{|X(i-1)|\times|X(i)|}$. Use $s$ to denote some $(i-1)$-face and $t$ to denote some $i$-face. Then,*

$$U_{i-1}[t,s] = \frac{1}{i}\mathbf{1}_{s\subseteq t}, \quad D_i[s,t] = \frac{w_i(t)}{w_{i-1}(s)}\mathbf{1}_{s\subseteq t}.$$

$$U_{i-1}D_i[t,t'] = \sum_{s\subseteq t \text{ and } t'} \frac{1}{(i+1)} \cdot \frac{w_i(t')}{w_{i-1}(s)}, \quad D_i U_{i-1}[s,s'] = \sum_{t\supseteq s \text{ and } s'} \frac{w_i(t)}{w_{i-1}(s)} \cdot \frac{1}{(i+1)}.$$

From the definition we make the following observations.

1. $U_{-1}D_0 = \vec{1}(\pi_0)^\top$.
2. $D_1 U_0 = \frac{1}{2}A + \frac{1}{2}Id$.
3. $\langle D_i f, g\rangle = \langle f, U_{i-1} g\rangle$.

The spectrum of these operators are related as follows.

**Claim 10.6.**

$$\mathrm{Spec}(U_{i-1}D_i) = \mathrm{Spec}(D_iU_{i-1}) = \mathrm{Spec}(U_{i-1})^2.$$

*Proof.* Let $g$ be an eigenfunction of $U_{i-1}D_i$:

$$U_{i-1}D_i g = \lambda g, \qquad \lambda \neq 0.$$

Then let $f = D_i g$. We get

$$D_iU_{i-1}f = D_iU_{i-1}D_i g = \lambda D_i g = \lambda f.$$

Thus,

$$\mathrm{Spec}(U_{i-1}D_i) = \mathrm{Spec}(D_iU_{i-1}).$$

Furthermore,

$$\|U_{i-1}f\|^2 = \langle U_{i-1}f, U_{i-1}f\rangle = \langle D_iU_{i-1}f, f\rangle = \lambda \|f\|^2$$

So,

$$\mathrm{Spec}(D_iU_{i-1}) = \mathrm{Spec}(U_{i-1})^2.$$

□

**Definition 10.7** ($i$-dimensional random walk)**.** *In a $d$-dimensional simplicial complex $X$, the $i$-dimensional* upper non-lazy random walk operator *is*

$$M_i^+ : C^i(X) \to C^i(X)$$

*defined by*

$$M_i^+ = \frac{i+2}{i+1}\left(D_{i+1}U_i - \frac{1}{i+2}Id\right) = \frac{i+2}{i+1}D_{i+1}U_i - \frac{1}{i+1}Id.$$

*In particular, $M_0^+ = A$. In general, $M_i^+$ is obtained by removing the self-loops in $D_{i+1}U_i$.*

**Lemma 10.8.** *If a $d$-dimensional simplicial complex $X$ is a one-sided (resp. two-sided) $\gamma$-expander, then for all $0 \leq i < d$,*

$$\lambda_1\left(M_i^+ - U_{i-1}D_i\right) \leq \gamma \quad \textit{(resp. } \|M_i^+ - U_{i-1}D_i\|_{\mathrm{op}} \leq \gamma \textit{ for two-sided).}$$

*Proof.* Decompose $f : X(i) \to \mathbb{R}$ into

$$f|_t : X_t(0) \to \mathbb{R}, \quad f|_t(u) = f(t \cup \{u\}), \quad \forall t \in X(i-1).$$

Then

$$\begin{aligned}\langle M_i^+ f, f\rangle &= \mathbb{E}_{t\sim\pi(i-1)}\, \mathbb{E}_{(u,v)\sim\pi_t(1)}\, f(t\cup\{u\})f(t\cup\{v\})\\ &= \mathbb{E}_{t\sim\pi(i-1)}\, \langle f|_t, A_t f|_t\rangle_t\,.\end{aligned}$$

Note that the first inequality holds because for all $t \in X(i-1)$ and $(u,v) \in X_t(1)$, on the LHS the coefficient for $f(t\cup\{u\})f(t\cup\{v\})$ is

$$\pi_i(t\cup\{u\}) \cdot \frac{w_{i+1}(t\cup\{u,v\})}{w_i(t\cup\{u\})} \cdot \frac{1}{i+1} + \pi_i(t\cup\{v\}) \cdot \frac{w_{i+1}(t\cup\{u,v\})}{w_i(t\cup\{v\})} \cdot \frac{1}{i+1} = \frac{2Z_i}{i+1} \cdot w_{i+1}(t\cup\{u,v\}),$$

where on the RHS the coefficient is

$$\pi_{i-1}(t) \cdot \frac{w_{i+1}(t \cup \{u,v\})}{\sum_{(u'v') \in X_t(1)} w'_{i+1}(t \cup \{u,v\})} = 2\pi_{i-1}(t) \cdot \frac{w_{i+1}(t \cup \{u,v\})}{w_{i-1(t)}} = 2Z_{i-1} \cdot w_{i+1}(t \cup \{u,v\}).$$

Since

$$Z_i^{-1} = \sum_{\sigma \in X(i)} w_i(\sigma) = \frac{1}{i+1} \sum_{t \in X(i-1)} w_{i-1}(t) = \frac{1}{i+1} \cdot Z_{i-1}^{-1},$$

the coefficients are equal and the equality holds.

We also consider

$$\langle U_{i-1} D_i f, f \rangle = \langle D_i f, D_i f \rangle.$$

Expanding,

$$\begin{aligned} \langle D_i f, D_i f \rangle &= \mathbb{E}_{t \sim \pi(i-1)} \Big( \mathbb{E}_{u \sim \pi_t(0)} f(t \cup \{u\}) \Big)^2 \\ &= \mathbb{E}_{t \sim \pi(i-1)} \langle f|_t, f|_t \rangle_t. \end{aligned}$$

Therefore,

$$\langle (M_i^+ - U_{i-1} D_i) f, f \rangle = \mathbb{E}_{t \sim \pi(i-1)} \langle (A_t - Id) f|_t, f|_t \rangle_t.$$

For any $t$, let $\bar{f}|_t = f|_t - \mathbb{E}_{u \sim \pi_t(0)}[f|_t(u)]$

$$\langle (A_t - Id) f|_t, f|_t \rangle_t = \langle A_t \bar{f}|_t, \bar{f}|_t \rangle_t \le \gamma \left\langle f|_t, f|_t \right\rangle_t.$$

Extending to all $t$, we obtain

$$\langle (M_i^+ - U_{i-1} D_i) f, f \rangle \le \gamma \, \mathbb{E}_{t \sim \pi(i-1)} \left\langle f|_t, f|_t \right\rangle_t = \gamma \langle f, f \rangle.$$

Thus,

$$\lambda_1 \left( M_i^+ - U_{i-1} D_i \right) \le \gamma.$$

The two-sided result is proven by also observing that

$$\langle (A_t - Id) f|_t, f|_t \rangle_t = \langle A_t \bar{f}|_t, \bar{f}|_t \rangle_t \ge -\gamma \left\langle f|_t, f|_t \right\rangle_t,$$

and deducing that

$$\lambda_{\min} \left( M_i^+ - U_{i-1} D_i \right) \ge -\gamma.$$

□

**Theorem 10.9** (Kaufman–Oppenheim)**.** *If $X$ is a $d$-dimensional one-sided $\gamma$-expander, then for all $-1 \le i < d$,*

$$\lambda_2(U_i D_{i+1}) \le \frac{i+1}{i+2} + (i+1)\gamma.$$

*Proof.* Recall that

$$\lambda_1(M_i^+ - U_{i-1}D_i) \leq \gamma I. \quad \text{and} \quad D_{i+1}U_i = \frac{1}{i+2}Id + \frac{i+1}{i+2}M_i^+.$$

Observe that the base is satisfied since $\lambda_2(U_{-1}D_0) = 0$. By induction if $\lambda_2(U_{i-1}D_i) \leq \frac{i}{i+1} + i\gamma$, then by Lemma 10.8

$$\lambda_2(M_i^+) \leq \lambda_2(U_{i-1}D_i) + \gamma \leq \frac{i}{i+1} + (i+1)\gamma,$$

which implies

$$\lambda_2(D_{i+1}U_i) = \lambda_2\left(\frac{1}{i+2}Id + \frac{i+1}{i+2}M_i^+\right) \leq \frac{i+1}{i+2} + (i+1)\gamma.$$

Finally, by the spectral equivalence between the two operators we conclude

$$\lambda_2(U_iD_{i+1}) \leq \max\left(\lambda_2(D_{i+1}U_i), 0\right) \leq \frac{i+1}{i+2} + (i+1)\gamma.$$

□

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# Lecture 12: Matroid Bases Sampling and HDXes

*Instructor: Siqi Liu* — *Scribe: Siqi Liu*

## 12.1 Application 1: matroids basis sampling

The first application of HDXes is matroid basis sampling. In this application we actually prove that a class of natural random walks can be viewed as random walks between faces of some implicit simplicial complexes. Thus to understand the mixing time of such random walks, it suffices to understand the expansion property of the corresponding simplicial complexes.

**Definition 12.1** (Matroid)**.** *A* matroid *is a pair $M = (V, \mathcal{I})$ where $V$ is a finite ground set and $\mathcal{I} \subseteq 2^V$ is a collection of subsets of $V$, called the* independent sets*, satisfying:*

*(1)* ***Hereditary property:*** *If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.*

*(2)* ***Exchange property:*** *If $I, J \in \mathcal{I}$ with $|J| < |I|$, then there exists $v \in I \setminus J$ such that $J \cup \{v\} \in \mathcal{I}$.*

*The* rank *of the matroid is the size of any maximal independent set (called a* basis*).*

**Example 12.2** (Spanning tree matroid)**.** *Let $G = (V, E)$ be a connected graph. The ground set is $E$, and $\mathcal{I}$ is the collection of edge sets that form forests in $G$. The bases of this matroid are the spanning trees of $G$.*

**Example 12.3** (Linear (or vector) matroid)**.** *Let $A \in \mathbb{F}_q^{n\times m}$ be a matrix over a field $\mathbb{F}_q$. The ground set is the set of columns of $A$, and $\mathcal{I}$ is the collection of linearly independent subsets of columns. The rank of this matroid is the rank of the matrix $A$.*

**Example 12.4** (Uniform matroid)**.** *Let $V$ be an $n$-element set, and fix $k \le n$. The independent sets are all subsets of $V$ of size at most $k$. This is called the uniform matroid $U_{k,n}$.*

**Lemma 12.5.** *Given a rank-$d$ matroid $M = (V, \mathcal{I})$, construct from it the $(d-1)$-dimensional simplicial complex $X$ by imposing the uniform distribution over the bases of $M$. Then $X$ is a $(d-1)$-dimensional one-sided $0$-expander.*

*Proof.* We prove the statement using the trickle-down theorem.

First show link connectedness: give a set $\sigma$ of rank $k \le d-2$, by the exchange property there exists some rank $k+1$ set $\upsilon$ containing $\sigma$. Furthermore by the hereditary property, $\upsilon$ has two subsets $\tau_1, \tau_2 \in M$ that both contain $\sigma$. So there is an edge connecting $\tau_1$ and $\tau_2$ in the link of $\sigma$.

For any other rank $k+1$ set $\tau_3$ in the link of $\sigma$, since $|\tau_3| < |\upsilon|$, by the exchange property there exists some element $e \in \upsilon$ s.t. $\tau_3 \cup \{e\}$ is a rank $k+1$ set in $M$. Because $e \in \upsilon$, so $\tau_3 \cup \{e\}$ contains either $\tau_1$ or $\tau_2$. Assume it contains $\tau_1$, then there is an edge connecting $\tau_1$ and $\tau_3$ in the link of $\sigma$.

In fact this implies that the link of $\sigma$ is a complete multipartite graph where each partition is given by $\sigma \times V_i$ where the $v_i$'s in $V_i$ cannot coexists in $M$. This is best seen through examples.

**Examples.**

- In the spanning tree case, if $\sigma$ is a forest with $n-k$ connected components over $n$ vertices, then its link corresponds to a complete $\binom{n-k}{2}$-partite graph (some parts may be empty) where each set of the matrices. $\binom{n-k}{2}$ count the parts each consisting of edges connecting two connected component $i, j$ in $\sigma$.
- In the linear matroid case, the rank $(k+1)$ sets in the link of a rank $k$ set $\sigma$ are partitions into $\frac{q^{n-k}-1}{q-1}$ equivalence classes where in each class all the sets have the same span.

So we see that all the links are connected.

Furthermore, the rank $(d-2)$ sets' links are always complete multipartite graphs (tripartite in the spanning tree matroids and $\frac{q^{n-d+2}-1}{q-1}$-partite in linear matroids). Therefore the 1-skeleton of the link is a one-sided 0-expander. Applying the trickle-down theorem we conclude that the simplicial complex is a $(d-1)$-dimensional one-sided 0-expander. □

**Theorem 12.6** (Anari–Liu–Oveis Gharan–Vinzant). *Given a matroid $M$ with $n$ ground set elements and rank $r$, the mixing time of the lazy down-up walk $\frac{r+1}{r+2}U_{r-2}D_{r-1} + \frac{1}{r+2}Id$ is*

$$t_\varepsilon = O\big(r^2 \log \tfrac{n}{\varepsilon}\big)\,.$$

*Proof.* We have the number of rank $r$ sets in $M$ is $N \le n^r$, and the second largest eigenvalue in absolute value of $\frac{r+1}{r+2}U_{r-2}D_{r-1} + \frac{1}{r+2}Id$ is $\gamma = \frac{1}{r+2}$. Hence $\log\frac{1}{\gamma} = \Theta\big(\frac{1}{r}\big)$.

So the mixing time is

$$t_\varepsilon = O\Big(\frac{\log(N/\varepsilon)}{\log(1/\gamma)}\Big) = O\big(r^2 \log \tfrac{n}{\varepsilon}\big)\,.$$

□

## 12.2 Application 2: agreement test

One of the main applications of high dimensional expanders is in the construction of *agreement tests*, which play a central role in hardness of approximation.

### 12.2.1 Definition of agreement tests

Let the *ground set* be denoted $V$. We consider a collection of subsets

$$T = \{s : s \subseteq V\}.$$

Let $\Sigma$ be an alphabet and $\{f_s : s \to \Sigma\}_{s\in T}$ be a collection of local functions.

An *agreement test* is defined by $V, T$, and a distribution $D$ over pairs of subsets $(s_1, s_2) \in T \times T$. Given as input a collection of local functions $\{f_s : s \to \Sigma\}_{s\in T}/$ The test samples a pair $(s_1, s_2)$ from $D$ and *accepts* if

$$f_{s_1}|_{s_1\cap s_2} \equiv f_{s_2}|_{s_1\cap s_2}.$$

The goal of an agreement test is to check whether there exists a global function $g : V \to \Sigma$ with which most of the local functions in the input agree. More precisely a $\kappa$-agreement test should satisfy:

For all $\{f_s\}_{s \in T}$,

$$\kappa \cdot \Pr_{(s_1,s_2)\sim D}[\text{test rejects on } (s_1, s_2)] \;\; \geq \;\; \text{dist}(\{f_s\}),$$

where $\text{dist}(\{f_S\}) = \min_{g:V\to\Sigma} \Pr_{s_1\sim D}[f_{s_1} \neq g|_{s_1}]$.

This condition ensures that the test rejects all far-from-consistent input with good probability.

## 12.3 Direct product tests

The agreement test described earlier can be implemented in different ways. The simplest version is the *direct product test*, where subsets are chosen uniformly at random from $\binom{V}{k}$.

### The direct product test

Fix a universe $[n] = \{1, 2, \ldots, n\}$, and consider subsets of size $k$. The direct product test operates as follows:

(i) Sample a random subset $t$, of size $k/2$.

(ii) Then independently sample two $s_1, s_2$ of size $k$ uniformly conditioned on containing $t$

(iii) Accept if the two functions agree on their intersection:

$$f_{s_1}|_{s_1\cap s_2} \equiv f_{s_2}|_{s_1\cap s_2}.$$

A result by Dinur and Steurer showed that the direct product tests are agreement tests.

**Theorem 12.7** (Dinur–Steurer)**.** *Let $n$ be large and $k$ moderately sized. For any collection of local functions $\{f_s : s \to \Sigma\}$ on subsets of size $k$, if the direct product test rejects with probability $\varepsilon$, then there exists a global function $g : [n] \to \Sigma$ such that*

$$\Pr_{s\sim Uniform(\binom{[n]}{k})}[f_s \not\equiv g|_s] \leq O(\varepsilon).$$

In other words, local consistency across random subsets implies the existence of a near-global assignment.

The direct product test has two drawbacks:

(a) Sampling from the distribution $DP$ requires too much random bits.

(b) The collection of subsets $T$ is too big for efficient implementation.

Next time we will see how this construction can be drastically improved by considering agreement tests constructed from HDXes.

**590.05 Expanders and High-dimensional Expanders** **Fall 2025**

# Lecture 13: Agreement Tests on High-dimensional Expanders

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 13.1 Agreement tests from high-dimensional expanders

The above result was generalized to richer structures. Instead of working with all subsets of size $k$, one can consider $(k-1)$-faces of high-dimensional expanders.

**Theorem 13.1** (Dinur–Kaufman)**.** *Fix $k \in \mathbb{N}$, for any $d > (k+1)^2$ and $\lambda \ll \frac{1}{d^2}$, the following test $(X(0), X(k), D)$ on a $d$-dimensional two-sided $\lambda$-expander $X$ equipped with the uniform distribution over $X(k)$ and $X(0)$ is a $\alpha$-agreement test for some $\alpha \geq 1$.*

1. *Sample a $d$-face $a \sim \pi(d)$ of $X$.*
2. *Inside a sample $(s_1, s_2) \sim DP(a)$ from the direct product test over $a$.*

This construction overcome the limitations of the direct product tests and has been used to improve parameters of the PCP theorem. This boils down to proving the following three lemmas.

First we discuss how to find the global function $g$ from the partial functions $\{f_s\}_{s \in X(k)}$:

1. For each $a \in X(d)$, find the function $g_a : a \to \Sigma$ to be
$$g_a(x) = \text{Maj}\{f_s(x) \mid x \in s \subset a\}.$$
2. From the partial functions $g_a$ reconstruct $g : X(0) \to \Sigma$ as
$$g(x) = \text{Maj}\{g_a(x) \mid x \in a \ \wedge \ a \in X(d)\}.$$

**Lemma 13.2.** *Assume $X$ satisfies the conditions in Theorem 13.1. If the test fails with probability $\varepsilon$, then*
$$\Pr_{x \sim \pi(0)}[\Pr_{s \sim \pi(k) | s \ni x}[f_s(x) \neq g(x)] \geq 2/5\ ] \leq O\left(\frac{\varepsilon}{k}\right).$$

**Lemma 13.3.** *Assume $X$ satisfies the conditions in Theorem 13.1. For any $i < j \leq d$,*
$$|\lambda|_2\left(D_{i+1} D_{i+2} \dots D_j U_{j-1} U_{j-2} \dots U_i\right) \leq \frac{i+1}{j+1} + O((j^2\lambda).$$

**Lemma 13.4.** *Let $G = (L, R, E)$ be a biregular bipartite graph with normalized biadjacency matrix $B$ whose second largest singular value is $\sigma$. Prove the following concentration result. For any subset $S \subset L$ s.t. $\frac{|S|}{|L|} = \varepsilon$ and any $C > 0$,*
$$\Pr_{v \sim \text{Uniform}(R)}\left[\frac{|N(v) \cap S|}{|N(v)|} > (1+C) \cdot \varepsilon\right] < \frac{\sigma^2}{\varepsilon C^2}\ .$$

Given these three lemmas we can prove the theorem as follows.

*Proof of Theorem 13.1.* Our goal is to upper bound the distance from $\{f_s\}$ to $g$ using the disagreement probability. We start by a triangle inequality,

$$\begin{aligned}\Pr_{s\sim\pi(k)}[f_s \not\equiv g|_s] &\le \Pr_{a\sim\pi(d),s\subset a}[f_s \not\equiv g_a|_s \ \vee \ g_a|_s \not\equiv g|_s] \\ &\le \Pr_{a\sim\pi(d),s\subset a}[f_s \not\equiv g_a|_s] + \Pr_{a\sim\pi(d),s\subset a}[g_a|_s \not\equiv g|_s]\end{aligned}$$

To bound the first term, we observe that it is the expected distance from a collection of smaller direct product test with agreement test parameter $\alpha_0$. Therefore

$$\Pr_{a\sim\pi(d),s\subset a}[f_s \not\equiv g_a|_s] = \mathbb{E}_{a\sim\pi(d)}\left[\Pr_{s\subset a}[f_s \not\equiv g_a|_s]\right] \le \mathbb{E}_{a\sim\pi(d)}\left[\alpha\cdot \Pr_{(s_1,s_2)\sim DP(a)}[f_{s_1}|_{s_1\cap s_2} \not\equiv f_{s_2}|_{s_1\cap s_2}]\right] = \alpha_0\varepsilon.$$

Recall that $DP(a)$ denotes the distribution given by the direct product test over $a$.

To bound the second term,

$$\begin{aligned}\Pr_{a\sim\pi(d),s\subset a}[g_a|_s \not\equiv g|_s] &\le \mathbb{E}_{a\sim\pi(d),s\subset a}\left[\sum_{x\in s}\mathbf{1}[g_a(x)\ne g(x)]\right] \\ &\le (k+1)\cdot\mathbb{E}_{a\sim\pi(d),s\subset a}\left[\Pr_{x\in s}[g_a(x)\ne g(x)]\right] \\ &= (k+1)\cdot\mathbb{E}_{x\sim\pi(0)}\left[\Pr_{a\sim\pi(d)|a\ni x}[g_a(x)\ne g(x)]\right]\end{aligned}$$

We bound the expectation separately for two different types of $x$'s. We call an $x$ good if $\Pr_{s\sim\pi(k)|s\ni x}[f_s(x)\ne g(x)] < 2/5$. Otherwise we say that $x$ is bad. Then we can bound the expectation by

$$\begin{aligned}&= (k+1)\cdot\left(\Pr_x[x \text{ is bad}] + \Pr_x[x \text{ is good}]\cdot\mathbb{E}_x[\Pr_{a\sim\pi(d)|a\ni x}[g_a(x)\ne g(x)] \mid x \text{ is good}]\right) \\ &= O(\varepsilon) + (k+1)\cdot\Pr_x[x \text{ is good}]\cdot\mathbb{E}_x[\Pr_{a\sim\pi(d)|a\ni x}[g_a(x)\ne g(x)] \mid x \text{ is good}],\end{aligned}$$

where the last step follows from Lemma 13.2.

For any **good** $x$, we bound $\Pr_{a\sim\pi(d)|a\ni x}[g_a(x)\ne g(x)]$ by considering the bipartite graph $B_x$ between the $(k-1)$-faces (equivalently all $k$-faces containing $x$) and $(d-1)$-faces (equivalently all $d$-faces containing $x$) in the link of $x$.

Let $S = \{s\ni x \mid f_s(x)\ne g(x)\}$ be a subset of vertices in $B_x$. For any $a$ such that $g_a(x)\ne g(x)$, at least half of the $k$-faces in $N(a)$ must be in $S$ so that their majority vote $g_a(x)\ne g(x)$. So the conditional probability can be upper bounded by

$$\begin{aligned}\Pr_{a\sim\pi(d)|a\ni x}[g_a(x)\ne g(x)] &\le \mathbb{E}_{x\sim\pi(0)}\left[\Pr_{a\sim\pi(d)|a\ni x}\left[\frac{|N(a)\cap S|}{|N(a)|}\ge(1+1/10\pi_x(S))\cdot\pi_x(S)\right]\right] \\ &\le \mathbb{E}_{x\sim\pi(0)}[100\pi_x(S)\sigma(B_x)^2] \\ &= O\left(\frac{1}{k}\right)\Pr_{s\sim\pi(k)|s\ni x}[f_s(x)\ne g(x)]\end{aligned}$$

The second inequality follows from Lemma 13.4 and $\sigma(B_x)$ denotes the second largest singular value of the normalized biadjacency matrix of $B_x$. The last equality is obtained by plugging in the value of $\sigma(B_x)$ given by Lemma 13.3. Thereby we can bound the contribution from good $x$'s by

$$(k+1)\cdot\Pr_x[x \text{ is good}]\cdot\mathbb{E}_x\left[\Pr_{a\sim\pi(d)|a\ni x}[g_a(x)\ne g(x)] \mid x \text{ is good}\right] \le O(1)\cdot\mathbb{E}_x\left[\Pr_{s\sim\pi(k)|s\ni x}[f_s(x)\ne g(x)]\cdot\mathbf{1}[x \text{ is good}]\right].$$

This average probability of disagreement at a random $x$ can be bounded by considering the bipartite graph between the $k$-faces containing $x$ and the $k/2$-faces contain $x$. Let $G_x$ denote the two-step walk graph over this bipartite graph starting and ending on the $k$-faces and with transition probability given by the operator $U_{k-2} \dots U_{k/2-1} D_{k/2} D_{k-1}$ in $X_x$ the link of $x$.

Define a vertex subset $S = \{s \mid f_s(x) \neq g(x)\}$, then

$$\begin{aligned}
\mathbb{E}_x \left[ \Pr_{s \sim \pi(k) | s \ni x} [f_s(x) \neq g(x)] \cdot \mathbf{1}[x \text{ is good}] \right] &= \mathbb{E}_x \left[ \pi_{G_x}(S) \cdot \mathbf{1}[\pi_{G_x}(S) < 2/5] \right] \\
&\leq \mathbb{E}_x \left[ \frac{\Pr_{\{s_1,s_2\} \sim E(G_x)} [\{s_1, s_2\} \in E(S, \overline{S})]}{\Phi(G_x)} \right] \\
&\leq O(1) \cdot \mathbb{E}_{x \sim \pi(0)} \left[ \Pr_{(s_1,s_2) \sim D | x \in s_1 \cap s_2} [f_{s_1}(x) \neq f_{s_2}(x)] \right] \\
&\leq O(1) \cdot \Pr_{(s_1,s_2) \sim D} [f_{s_1}|_{s_1 \cap s_2} \not\equiv f_{s_2}|_{s_1 \cap s_2}] = O(\varepsilon).
\end{aligned}$$

The first inequality holds by definition of conductance of a set, and the second inequality is a consequence of $G_x$ have bounded second eigenvalue (by Lemma 13.3) and thus constant graph conductance.

Combine this upper bound with the previous ones, we conclude that there exists some $\alpha > 0$ such that

$$\Pr_{s \sim \pi(k)} [f_s \not\equiv g|_s] \leq \alpha \varepsilon.$$

□

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# Lecture 14: Agreement Tests on High-dimensional Expanders (cont.)

*Instructor: Siqi Liu* — *Scribe: Siqi Liu*

Now we are left to prove the following lemmas.

**Lemma 14.1** (Restating Lemma 13.2)**.** *Assume $X$ satisfies the conditions in Theorem 13.1. If the test fails with probability $\varepsilon < \frac{1}{10\alpha_0}$ ($\alpha_0$ being the agreement test parameter for direct product tests over d-faces), then*

$$\Pr_{x\sim\pi(0)}\left[\Pr_{s\sim\pi(k)|s\ni x}[f_s(x)\neq g(x)]\geq 2/5\right]\leq O\left(\frac{\varepsilon}{k}\right).$$

**Lemma 14.2** (Restating Lemma 13.3)**.** *Assume $X$ satisfies the conditions in Theorem 13.1. For any $i<j\leq d$,*

$$|\lambda|_2\left(D_{i+1}D_{i+2}\dots D_jU_{j-1}U_{j-2}\dots U_i\right)\leq\frac{i+1}{j+1}+O(j^2\lambda).$$

**Lemma 14.3** (A variant of Lemma 13.4)**.** *Let $G=(L,R,E)$ be a bipartite graph with normalized biadjacency matrix $B$ whose second largest singular value is $\sigma$. For any function $h:L\to[0,1]$ s.t. $\mathbb{E}_{u\in L}[h(u)]=\mu$ and any $C>0$,*

$$\Pr_{v\sim(R)}\left[|Bh(v)-\mu|>C\cdot\mu\right]<\frac{\sigma^2}{C^2\mu}\ .$$

*Proof of Lemma 14.1.* Following the convention of last lecture, we say that $x\in X(0)$ is bad if $\Pr_{s\sim\pi(k)|s\ni x}[f_s(x)\neq g(x)]\geq 2/5$. We first show that for any bad $x$, most of the $d$-faces $a\ni x$ has its local majority $g_a(x)$ not being a vast majority. More precisely, we prove for any bad $x$

$$\Pr_{a\sim\pi(d)|a\ni x}\left[\Pr_{s\sim\pi(k)|a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}\right]\geq 1-O\left(\frac{1}{k}\right). \tag{14.1}$$

This is again due to the expansion of the bipartite graph $B_x$ between the $(k-1)$-faces (equivalently all $k$-faces containing $x$) and $(d-1)$-faces (equivalently all $d$-faces containing $x$) in the link of $x$.

Again let $S=\{s\ni x\mid f_s(x)\neq g(x)\}$ be a subset of vertices in $B_x$. Note that for any $d$-face $a\ni x$,

$$\Pr_{s\sim\pi(k)|a\supset s\ni x}[g_a(x)\neq f_s(x)]\leq\Pr_{s\sim\pi(k)|a\supset s\ni x}[g(x)\neq f_s(x)].$$

Therefore we have

$$\begin{aligned}\Pr_{a\sim\pi(d)|a\ni x}\left[\Pr_{s\sim\pi(k)|a\supset s\ni x}[g_a(x)\neq f_s(x)]<\frac{3}{10}\right]&\leq\Pr_{a\sim\pi(d)|a\ni x}\left[\Pr_{s\sim\pi(k)|a\supset s\ni x}[g(x)\neq f_s(x)]<\frac{3}{10}\right]\\&=\Pr_{a\sim\pi(d)|a\ni x}\left[\frac{|N(a)\cap S|}{|N(a)|}<\frac{3}{10}\right]\\&\leq\Pr_{a\sim\pi(d)|a\ni x}\left[\frac{|N(a)\cap S|}{|N(a)|}<(1-1/10\pi_x(S))\cdot\pi_x(S)\right]\\&=\frac{1}{k}\cdot\Pr_{s\sim\pi(k)|s\ni x}[f_s(x)\neq g(x)]=O\left(\frac{1}{k}\right)\end{aligned}$$

The last line follows from the sampling property of expanders and that the second largest singular value $\sigma(B_x) \leq \sqrt{\frac{1}{k}}$. Thereby we obtain Equation (14.1).

Now we proceed to prove the statement by contradiction. Assume that

$$\Pr_{x\sim\pi(0)}\left[\Pr_{s\sim\pi(k)|s\ni x}[f_s(x)\neq g(x)]\geq 2/5\right]\geq\omega\left(\frac{\varepsilon}{k}\right).$$

Then by Equation (14.1)

$$\Pr_{x\sim\pi(0),a\sim\pi(d)|a\ni x}\left[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}\right]\geq\omega\left(\frac{\varepsilon}{k}\right)\cdot\left(1-O\left(\frac{1}{k}\right)\right)=\omega\left(\frac{\varepsilon}{k}\right).$$

Furthermore denote $\varepsilon_a = \Pr_{(s_1,s_2)\sim DP(a)}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}]$ and deduce that

$$\begin{aligned}
&\Pr_{x\sim\pi(0),a\sim\pi(d)|a\ni x}\left[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}\right]\\
=&\mathbb{E}_{a\sim\pi(d)}\left[\Pr_{x\in a}[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}]\mid\varepsilon_a>\varepsilon+\frac{1}{10\alpha_0}\right]\cdot\Pr_{a\sim\pi(d)}\left[\varepsilon_a>\varepsilon+\frac{1}{10\alpha_0}\right]\\
&+\mathbb{E}_{a\sim\pi(d)}\left[\Pr_{x\in a}[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}]\mid\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}\right]\cdot\Pr_{a\sim\pi(d)}\left[\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}\right]\\
\leq&\Pr_{a\sim\pi(d)}\left[\varepsilon_a>\varepsilon+\frac{1}{10\alpha_0}\right]+\mathbb{E}_{a\sim\pi(d)}\left[\Pr_{x\in a}[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}]\mid\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}\right]\cdot\Pr_{a\sim\pi(d)}\left[\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}\right]
\end{aligned}$$

So either

$$\Pr_{a\sim\pi(d)}\left[\varepsilon_a>\varepsilon+\frac{1}{10\alpha_0}\right]=\omega\left(\frac{\varepsilon}{k}\right), \tag{14.2}$$

or

$$\mathbb{E}_{a\sim\pi(d)}\left[\Pr_{x\in a}[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}]\mid\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}\right]\cdot\Pr_{a\sim\pi(d)}\left[\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}\right]=\omega\left(\frac{\varepsilon}{k}\right). \tag{14.3}$$

However, Equation (14.2) cannot hold by considering the bipartite graph between $X(2k)$ and $X(d)$ with normalized biadjacency matrix $B = U_{d-1}\dots U_{2k}$. Let $h : X(2k)\to\mathbb{R}$ be such that

$$h(\tau)=\Pr_{(s_1,s_2)\sim DP(\tau)}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}].$$

Note that $\varepsilon_a = Bh(a)$ and $\mathbb{E}_{\tau\sim\pi(2k)}[h(\tau)] = \varepsilon$.

Let $\vec{1}_{\geq\varepsilon+\frac{1}{10\alpha_0}}$ be the indicator function of all $a\in X(d)$ with $\varepsilon_a\geq\varepsilon+\frac{1}{10\alpha_0}$. Then by Lemma 14.3 we have that

$$\begin{aligned}
\Pr_{a\sim\pi(d)}\left[\varepsilon_a>\varepsilon+\frac{1}{10\alpha_0}\right]&\leq\Pr_{a\sim\pi(d)}\left[|Bh(a)-\varepsilon|>\frac{1}{10\alpha_0}\right]\\
&\leq 100\cdot\sigma(B)^2\varepsilon\\
&=O\left(\frac{\varepsilon}{k}\right).
\end{aligned}$$

Furthermore Equation (14.3) also cannot hold by considering for every $a\in X(d)$ with $\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}$ the bipartite graph between $k$-faces in $a$ and 0-faces in $a$ with the normalized biadjacency matrix $B_a = D_1\dots D_k$. Let $S=\{s\subset a\mid g_a|_s\not\equiv f_s\}$. Note that for any $x\in a$

$$\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}\Rightarrow\Pr_{a\supset s\ni x}[g_a|_s\not\equiv f_s]\geq\frac{3}{10}.$$

Therefore,

$$\Pr_{x\in a}\left[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}\right]\leq\Pr_{x\in a}\left[\Pr_{a\supset s\ni x}[g_a|_s\not\equiv f_s]\geq\frac{3}{10}\right]$$
$$=\Pr_{x\in a}\left[\frac{|N(x)\cap S|}{|N(x)|}-\Pr_{s\in a}[s\in S]\geq\frac{3}{10}-\Pr_{s\in a}[s\in S]\right]$$

When $\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}$,

$$\Pr_{s\in a}[s\in S]=\Pr_{s\in a}[g_a|_s\not\equiv f_s]\leq\alpha_0\cdot\Pr_{(s_1,s_2)\sim DP(a)}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}]=\alpha_0\cdot\varepsilon_a<\alpha_0\left(\frac{1}{10\alpha_0}+\frac{1}{10\alpha_0}\right)=\frac{1}{5}.$$

So for such $a$'s we can simplify the previous inequality to

$$\Pr_{x\in a}\left[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}\right]\leq\Pr_{x\in a}\left[\left|\frac{|N(x)\cap S|}{|N(x)|}-\Pr_{s\in a}[s\in S]\right|\geq\frac{3}{10}-\frac{1}{5}\right]$$
$$\leq 100\cdot\sigma(B_a)^2\cdot\Pr_{s\in a}[s\in S]=O_{\alpha_0}\left(\frac{1}{k}\right)\cdot\Pr_{(s_1,s_2)\sim DP(a)}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}]$$

where the last line is due to Lemma 14.3.

Furthermore

$$\mathbb{E}_{a\sim\pi(d)}\left[\Pr_{(s_1,s_2)\sim DP(a)}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}]\right]=\Pr_{(s_1,s_2)\sim D}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}]=\varepsilon.$$

As a result

$$\mathbb{E}_{a\sim\pi(d)}\left[\Pr_{x\in a}[\Pr_{a\supset s\ni x}[g_a(x)\neq f_s(x)]\geq\frac{3}{10}]\mid\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}\right]\cdot\Pr_{a\sim\pi(d)}\left[\varepsilon_a\leq\varepsilon+\frac{1}{10\alpha_0}\right]$$
$$\leq\mathbb{E}_{a\sim\pi(d)}\left[O_{\alpha_0}\left(\frac{1}{k}\right)\cdot\Pr_{(s_1,s_2)\sim DP(a)}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}]\right]$$
$$=O_{\alpha_0}\left(\frac{1}{k}\right)\cdot\mathbb{E}_{a\sim\pi(d)}\left[\Pr_{(s_1,s_2)\sim DP(a)}[f_{s_1}|_{s_1\cap s_2}\not\equiv f_{s_2}|_{s_1\cap s_2}]\right]$$
$$=O_{\alpha_0}\left(\frac{\varepsilon}{k}\right).$$

We derived a contradiction and thus complete the proof. □

*Proof of Lemma 14.2.* Let $f$ be the eigenfunction for the second largest eigenvalue in absolute value of $D_{i+1}D_{i+2}\ldots D_jU_{j-1}U_{j-2}\ldots U_i$. Note that $\left\langle f,\vec{1}\right\rangle_{\pi(i)}=0$. We can rewrite

$$|\lambda|_2\left(D_jU_{j-1}U_{j-2}\ldots U_i\right)\|f\|^2_{\pi(i)}=\langle D_jU_{j-1}U_{j-2}\ldots U_if,f\rangle_{\pi(i)}=\|U_{j-1}\ldots U_if\|^2_{\pi(j)}$$
$$\leq\sigma_2(U_{j-1})^2\left\|U_{j-2}\ldots U_if\right\|^2_{\pi(j-1)},$$

where $\sigma_2(U_{j-1})$ is the second largest singular value in absolute value of $U_{j-1}$. The second inequality holds because $\left\langle U_{j-2}\ldots U_if,\vec{1}\right\rangle_{\pi(j-1)}=\left\langle f,\vec{1}\right\rangle_{\pi(i)}=0$ and so $U_{j-2}\ldots U_if$ is orthogonal to the singular vector $\vec{1}$ of the largest singular value of $U_{j-1}$. Apply this step iteratively to get

$$|\lambda|_2\left(D_jU_{j-1}U_{j-2}\ldots U_i\right)\|f\|^2_{\pi(i)}\leq\sigma_2(U_{j-1})^2\cdot\cdots\cdot\sigma_2(U_i)^2\|f\|^2_{\pi(i)}.$$

From earlier results we have shown on eigenvalues of $D_{i+1}U_i$ we know that

$$\sigma_2(U_i)^2 = |\lambda|_2 \left(D_{i+1}U_i\right) \leq \frac{i+1}{i+2} + (i-1)\lambda.$$

When $\lambda \ll \frac{1}{d} < \frac{1}{j}$, we have that

$$|\lambda|_2 \left(D_j U_{j-1} U_{j-2} \dots U_i\right) \leq \left(\frac{j}{j+1} + (j-2)\lambda\right) \dots \left(\frac{i+1}{i+2} + (i-1)\lambda\right) \leq \frac{i+1}{j+1} + \sum_{\ell=1}^{j-i-1} j^{\ell+1}\lambda^{\ell} = \frac{i+1}{j+1} + O\left(j^2\lambda\right).$$

□

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# Lecture 15: Constructions of Bounded-degree High-dimensional Expanders

*Instructor: Siqi Liu* *Scribe:*

In this lecture we will see a construction of bounded-degree one-sided HDXs using coset complexes of matrix groups. Then using the result from HW4 Q1, we can construct bounded-degree two-sided HDXs by taking skeletons of the one-sided HDXs.

This lecture largely follows a previous lecture given by Yotam Dikstein. That lecture note is available here `https://www.wisdom.weizmann.ac.il/~dinuri/courses/22-HDX/L7.pdf`.

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# Lecture 16: Introduction to Coboundary Expanders

*Instructor: Siqi Liu* — *Scribe: Siqi Liu*

## 16.1 Chain Complexes

Let $X = \bigcup_{i=-1}^{d} X(i)$ be a $d$-dimensional complex (not necessarily simplicial, for instance can also be polyhedral complexes). Define the $i$-cochains:

$$C^i = \{f : X(i) \to \mathbb{F}_2\}.$$

In general one can define the cochains over any ring, but for simplicity we focus on the case of $\mathbb{F}_2$. Consider the Hasse diagram between $X(i)$'s. The containment relation between faces in $X$ induces the coboundary and boundary maps

$$\delta_i : C^i \to C^{i+1}, \quad \delta_{i+1}^* : C^{i+1} \to C^i,$$

where $\delta_i g(t) = \sum_{e \subset t} g(e)$ and $\delta_{i+1}^* h(e) = \sum_{t \supset e} h(t)$.

**Claim 16.1.** *If $X$ is a simplicial complex, then for any $g \in C^i(\mathbb{F}_2)$, $\delta_{i+1}\delta_i g \equiv \mathbf{0}$.*

## 16.2 Coboundary expansion and graph conductance

Consider a 1-dimensional simplicial complex $X$ and its chain complex:

$$C^{-1} \xrightarrow{\delta_{-1}} C^0 \xrightarrow{\delta_0} C^1$$

We define the 0-th coboundary $B^0$ and the 0-th cocycle $Z^0$ of $X$ as:

$$B^0 = \text{im}(\delta_{-1}) = \{\vec{1}, \vec{0}\}, \quad Z^0 = \ker(\delta_0) = \text{span}\{\vec{1}_{\text{CC}_1}, \dots, \vec{1}_{\text{CC}_m}\},$$

where $CC_i$'s are connected components in $X$. From Claim 16.1 it is clear that $B^0 \subset Z^0$.

The conductance of $X$ is

$$\Phi(X) = \min_{f \in C^0 \setminus B^0} \frac{wt_1(E(f, \overline{f}))}{\min(wt_0(f), wt_0(\overline{f}))} = \min_{f \in C^0 \setminus B^0} \frac{\|\delta_0 f\|}{\|f - B^0\|},$$

where $\|\delta_0 f\| = \mathbb{E}_{e \sim \pi(1)}[\delta_0 f(e)]$ and $\|f - B^0\| = \min_{g \in B^0} \mathbb{E}_{v \sim \pi(0)}[\mathbf{1}(f(v) \neq g(v))]$. So a graph $X$ has conductance at least $\eta$ if

$$\min_{f \in C^0 \setminus B^0} \frac{\|\delta_0 f\|}{\|f - B^0\|} \geq \eta.$$

Notice that if $X$ is connected, i.e. $B^0 = Z^0$, then the conductance is strictly great than 0. In this case we say that there is no 0-dimensional holes in $X$. Here a 0-dimensional hole is a nonempty set of 0-faces $f : X(0) \to \mathbb{F}_2$ such that $\delta_0^* f \equiv 0$, i.e. its boundary vanishes.

This generalizes to the $i$-dimensional *coboundary expansion*. A chain complex $X$ is a $i$-dimensional $\eta$-coboundary expander if

$$\min_{f \in C^i \setminus B^i} \frac{\|\delta_i f\|}{\|f - B^i\|} \geq \eta.$$

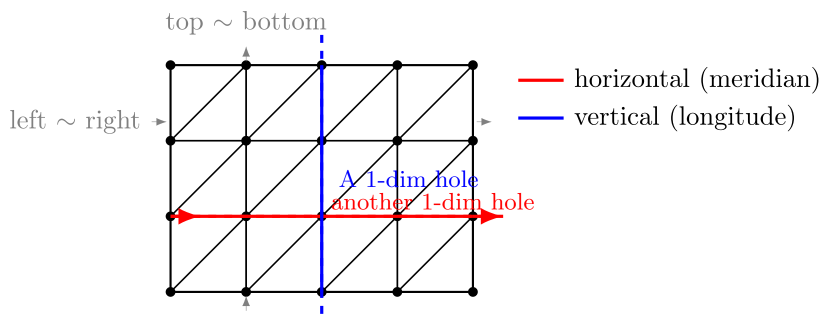
Again for $\eta > 0$, we at least need $B^i = Z^i$.

In a simplicial complex $X$, for $i = 1$:

$$B^1 = \{\text{edge cut}\},$$
$$Z^1 = \{g : X(1) \to \mathbb{F}_2 \mid \text{each triangle has an even number of edges in } g\}.$$

Thus, $B^1 = Z^1$ when there is no 1-dimensional hole. Again $g : X(1) \to \mathbb{F}_2$ is a 1-dimensional hole if $\delta_1^* g \equiv 0$, i.e. its boundary vanishes.

Let's see an example of a 2-dimensional complex with a 1-dimensional hole.




## 16.3 Coding-theoretic formulation

Let $A_{i-1}$ (resp. $A_i$) be the incidence matrices between $(i-1)$- and $i$-faces (resp. $i$- and $(i+1)$-faces of $X$) of $X$ . Then $\text{im}(A_{i-1})$ defines the code $B^i$. Define a local test for whether an $f$ is in $B^i$:

$$\text{Sample } \sigma \in X(i) \text{ uniformly, accept if } (A_i f)(\sigma) = 0.$$

Then $X$ is an $i$-dimensional $\eta$-coboundary expander iff this test is a good local test, i.e.

$$\text{dist}(f, B^i) \leq \eta^{-1} \cdot \Pr[\text{test rejects } f].$$

Since $\delta_i f = A_i f$, we have $\Pr[\text{test rejects } f] = \|\delta_i f\|$. If $X$ is an $i$-dimensional $\eta$-coboundary expander:

$$\text{im}(A_{i-1}) = \text{im}(\delta_{i-1}) = \ker(\delta_i) = \ker(A_i).$$

Hence $A_{i-1}$ is the generator matrix of $B^i$, and $A_i$ is the parity-check matrix.

**Remark 16.2.** *However these codes $B^i$ has very small distance and thus is not directly applicable in general.*

**Example 16.3.** *Known examples of $d$-dimensional coboundary expanders includes:*

- *Complete complexes (over all groups)*
- *Complete $(d+1)$-partite complexes (over all groups)*

- *Spherical buildings (over all groups)*
- *Ramanujan complexes from Lubotzky-Chapman (over finite groups)*
- *Kaufman-Oppenhiem coset complexes (over finite groups)*

## 16.4 Motivations

### Original motivation: Gromov's topological overlap property

Coboundary expansion is first defined by Gromov in order to understand what types complexes have the topological overlap property.

**Definition 16.4.** *A $d$-dimensional complex $X$ has the topological overlapping property if for any continuous map $\varphi : X(d) \to \mathbb{R}^d$, there exists a point $p \in \mathbb{R}^d$ contained in the interior of a constant fraction of $d$-faces of $X$.*

Gromov proved that this is implied by $i$-dimensional coboundary expansion of $X^{\leq i}$ for all $0 \leq i < d$. We can check the simple case of $i = 0$ and 1.

### Agreement tests in low-acceptance regimes

Recall that an *agreement test* on a $d$-dimensional simplicial complex $X$ is defined by $(X(0), X(k), D)$. Given as input a collection of local functions $\{f_s : s \to \Sigma\}_{s \in X(k)}$, the test samples a pair $(s_1, s_2)$ from $D$ and *accepts* iff

$$f_{s_1}|_{s_1 \cap s_2} \equiv f_{s_2}|_{s_1 \cap s_2}.$$

An agreement test in the low-acceptance regimes guarantees that there exists $\delta, \kappa > 0$ such that if the test accepts $\{f_s\}_{s \in X(k)}$ with probability at least $\delta$, then there exists a global function $g : X(0) \to \Sigma$ such that

$$\Pr_{s \sim D}[f_s = g|_s] \geq \kappa \cdot \Pr_{(s_1, s_2) \sim D}[f_{s_1}|_{s_1 \cap s_2} \equiv f_{s_2}|_{s_1 \cap s_2}].$$

This agreement test ensures that the if a collection of local functions is accepted with non-trivial probability, then there exists a global function that also agrees with a non-trivial fraction of the local functions. Agreement tests with this guarantee is very useful in hardness of approximation.

Recently, people have shown that the test on a simplicial complex $Y$ is an agreement test in the low-acceptance regime only if a certain chain complex related to $Y$ is a coboundary expander.

**Theorem 16.5** (Dikstein–Dinur, Bafna–Minzer)**.** *Let $X$ be a $d$-dimensional simplicial complex, and let $F_k(X)$ denote its face complex with*

$$F_k(X)(0) = X(k), \quad F_k(X)(2) = \{\{s_0, s_1, s_2\} \in X(k)^3 \mid s_0 \cup s_1 \cup s_2 \in X(3k+2)\}.$$

*If $F_k(X)$ is not a 1-dimensional coboundary expander, then the test over $X(k)$ is not an agreement test in the low-acceptance regime.*

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Lecture 17: The Local-to-global Theorem for Cosystolic Expanders

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## 17.1 Cosystolic Expansion

Constructing coboundary expanders is notoriously difficult, primarily because it is hard to design bounded-degree complexes that satisfy the condition $B^i = Z^i$. However, for many applications this requirement can be relaxed. Complexes that meet these weaker, yet still useful, expansion conditions are known as cosystolic expanders.

**Definition 17.1.** *A chain complex $X$ is a $i$-dimensional $\eta$-cosystolic expander if*

$$\min_{f \in C^i \setminus Z^i} \frac{\|\delta_i f\|}{\|f - Z^i\|} \geq \eta.$$

For some applications, it is also important to bound the weight of the functions in $Z^i \setminus B^i$. So we also define:

**Definition 17.2.** *A chain complex $X$ is a $i$-dimensional $(\mu, \eta)$-cosystolic expander if*

$$\min_{f \in C^i \setminus Z^i} \frac{\|\delta_i f\|}{\|f - Z^i\|} \geq \eta,$$

*and*

$$\forall f \in Z^i \setminus B^i, \quad \|f - B^i\| \geq \mu.$$

**Remark 17.3.** *Ramanujan complexes and many other coset complexes are known to be cosystolic expanders.*

**Applications.** Cosystolic expanders have found remarkable applications in various areas of math and computer science, including:

1. The **topological overlap property** (Kaufman–Kazhdan–Lubotzky).
2. Construction of **asymptotically optimal locally testable codes** and **quantum low-density parity-check (LDPC) codes** (Dinur–Evra–Livne–Lubotzky–Mozes; Panteleev–Kalachev; Leverrier–Zémor).
3. Establishing the **hardness of Max-$k$-XOR** for semidefinite programming (SDP) relaxations (Hopkins–Lin).

## 17.2 The local-to-global theorem for cosystolic expanders

**Theorem 17.4.** *Let $X$ be a 3-dimensional simplicial complex such that*

1. *for all $v \in X(0)$, the link $X_v$ is a 1-dimensional $\beta$-coboundary expander and*

2. *$X$ is a 3-dimensional $\gamma$-expander for $\gamma \ll \beta^2$,*

*then there exists $\eta = O(\beta)$ such that $X$ is a 1-dimensional $\eta$-cosystolic expander.*

To prove this theorem we need to first define locally minimal functions (cochains).

**Definition 17.5.** *A function $f \in C^i$ is locally minimal if for every $v \in X(0)$ and any localized function $h_v : X(1) \to \mathbb{F}_2$ such that $h_v(e) = 0,\ \forall v \notin e$, $f$ satisfies that*

$$\Pr_{e \sim X_v(1)}[\delta_1 f(e \cup \{v\}) \neq 0] \leq \Pr_{e \sim X_v(1)}[\delta_1 (f + h_v)(e \cup \{v\}) \neq 0].$$

*In other words, the induced local function $g_v : X_v(0) \to \mathbb{F}$ where $g_v(u) = f(\{u, v\})$ minimizes the following disagreement probability*

$$\Pr_{e \sim X_v(1)}\left[\delta_0 g_v(e) \neq f(e)\right].$$

**Remark 17.6.** *Notice that if we define the local function $f_v : X_v(1) \to \mathbb{F}_2$ to be $f_v(e) = f(e)$ and use $B^1_v$ to denote the 1-dimensional coboundary space of $X_v$, then*

$$\Pr_{e \sim X_v(1)}\left[\delta_0 g_v(e) \neq f(e)\right] = \left\| f_v - \delta_0 g_v \right\|.$$

*Now since $g_v$ is the minimizer we deduce that*

$$\left\| f_v - B^1_v \right\| = \left\| f_v - \delta_0 g_v \right\|.$$

The proof consists of two steps. First we start with some $f_0 \in C^i \setminus Z^i$, and we use an algorithm to "correct" $f_0$ into a locally minimal function $f$ iteratively while controlling the change in $\|\delta_1 f\|$ and $\left\| f - Z^1 \right\|$

**Step 1: Correcting $f_0$ to a locally minimal function** If there exists $v \in X(0)$ and $h_v : X(1) \to \mathbb{F}_2$ that is nonzero only for $e \ni v$, update $f = f + h_v$. Repeat this until no such $v, h_v$ exist. Note that this process terminates since in every iteration $\|\delta_1 f\|$ decreases by at least $\min_{t \in X(2)} wt_2(t)$. Also in every iteration the distance $\left\| f - Z^1 \right\|$ changes by as most $\max_{v \in X(0)} wt_0(v)$ since the update function $h_v$ is local.

If after $m$ iterations, $f \in Z^1$ (equivalently $\|\delta_1 f\| = 0$), then we know that

$$\|\delta_1 f_0\| \geq \sum_{i=1}^{m} \min_{t \in X(2)} wt_2(t) \quad \text{and} \quad \left\| f_0 - Z^1 \right\| \leq \|f_0 - f\| \leq \sum_{i=1}^{m} \max_{v \in X(0))} wt_0(v).$$

As a result for $f_0$

$$\frac{\|\delta_1 f_0\|}{\|f_0 - Z^1\|} \geq \frac{\min_{t \in X(2)} wt_2(t)}{\max_{v \in X(0))} wt_0(v)} = O(1).$$

Here the last equality holds for all bounded-degree simplicial complex with uniform distribution on their maximal faces. In fact, a more careful argument can remove this assumption and yield $O(\beta)$ using coboundary expansion of $X_v$'s. We omit it here and you will prove it in the next homework.

If at the end of this process $f \notin Z^1$, we consider step 2.

**Step 2: Prove that for all locally minimal $f$, $\|\delta_1 f\|$ is large** To prove this bound, we shall use the following two lemmas.

**Lemma 17.7.** *Consider an $X$ that satisfies the conditions in the theorem. For every locally minimal $f \in C^1 \setminus Z^1$, if $\Pr_{t\sim\pi(2)}[\delta_1 f(t) \neq 0] = \varepsilon \leq \frac{\beta}{6}$, then*

$$\Pr_{v\sim\pi(0)}\left[\Pr_{t\ni v}[\delta_1 f(t) \neq 0] \geq \frac{1}{5}\right] \leq \varepsilon \cdot O\left(\frac{\gamma}{\beta^2}\right) .$$

*Proof.* First define the set of bad vertices to be

$$V^* = \{v \in X(0) \mid \Pr_{t\ni v}[\delta_1 f(t) \neq 0] \geq \frac{1}{5}\}.$$

We define another related set

$$R = \{v \in X(0) \mid \Pr_{t\in X_v(2)}[\delta_1 f(t) \neq 0] \geq \frac{\beta}{5}\}.$$

We shall first show that $R \supseteq V^*$ and then bound the weight of $R$.

For every $v \in V^*$ define two local functions $g_v : X_v(0) \to \mathbb{F}_2$ and $f_v : X_v(1) \to \mathbb{F}_2$ as

$$g_v(a) = f(\{a, v\}), \quad f_v(\{a, b\}) = f(\{a, b\}).$$

By construction

$$\delta_0 g_v(\{a, b\}) \neq f_v(\{a, b\}) \Rightarrow f(\{v, a\}) + f(\{v, b\}) + f(\{a, b\}) \neq 0 \Rightarrow \delta_1 f(\{a, b, v\}) \neq 0.$$

Furthermore by local minimality, $\left\|f_v - B_v^1\right\| = \|f_v - \delta_0 g_v\| = \Pr_{t\ni v}[\delta_1 f(t) \neq 0]$. Since $X_v$ is a 1-dimensional $\beta$-coboundary expander,

$$\frac{\|\delta_1 f_v\|}{\|f_v - B_v^1\|} \geq \beta \Rightarrow \|\delta_1 f_v\| \geq \beta \cdot \Pr_{t\ni v}[\delta_1 f(t) \neq 0].$$

Observe that for every $t \in X_v(2)$ $\delta_1 f_v(t) = \sum_{e\subset t} f(e) = \delta_1 f(t)$. Thereby

$$\|\delta_1 f_v\| = \Pr_{t\in X_v(2)}[\delta_1 f(t) \neq 0].$$

Since $v \in V^*$, we have that

$$\Pr_{t\in X_v(2)}[\delta_1 f(t) \neq 0] \geq \beta \cdot \Pr_{t\ni v}[\delta_1 f(t) \neq 0] \geq \frac{\beta}{5}.$$

Therefore $v \in R$ as well.

We next bound the weight of $R$. Consider the bipartite graph $G$ between $X(0)$ and $X(2)$ where we connect $v$ and $t$ if and only if $\{v\} \cup t \in X(3)$. Use $\sigma$ to denote the second largest singular value of the normalized biadjacency matrix of $G$. Then by the bipartite expander sampling lemma we have that

$$\Pr_{v\sim\pi(0)}\left[\Pr_{t\in X_v(2)}[\delta_1 f(t) \neq 0] \geq \frac{\beta}{5}\right] \leq \varepsilon \cdot \frac{\sigma^2}{\left(\frac{\beta}{5} - \varepsilon\right)^2}$$

We thereby deduce that

$$wt_0(V^*) \leq wt_0(R) \leq \varepsilon \cdot \frac{\sigma^2}{(\beta/5 - \varepsilon)^2}.$$

By the next claim we know that $\sigma^2 \leq 12\gamma + O(\gamma^2)$. Thus we conclude the proof. $\square$

**Claim 17.8.** *The squared second largest singular value of the normalized biadjacency matrix of $G$ is bounded by $\sigma^2 \leq 12\gamma + O(\gamma^2)$.*

*Proof sketch.* Define the swap walk operator $S : C^1 \to C^1$ such that for any function $h \in C^1$

$$Sh(e) = \mathbb{E}_{\tau \sim \pi(3) | \tau \supset e}[h(\tau \setminus e)].$$

Essentially it encodes the random walk from an edge $e \in X(1)$ to a random $\tau \in X(3)$ containing and then deterministically move to the edge $\tau \setminus e$.

Recall that $G$ is a bipartite graph with edge weight $\{v, t\} = wt_3(t \cup \{v\})$. If we use $B$ to denote the 2-step random walk from $X(0)$ to $X(0)$ in $G$, then we claim that

$$B = 2 \cdot \left( D_1 S^2 U_0 - \frac{1}{2} \cdot D_1 S U_0 \right),$$

where $D_1$ and $U_0$ are down and up operators in $X$. Therefore to know $\sigma$, we study the second largest eigenvalue of $S$.

We note another operator decomposition involving $S$

$$D_2 D_3 U_2 U_1 = \frac{1}{2} D_2 U_1 + \frac{1}{3} M_1^+ + \frac{1}{6} S. \tag{17.1}$$

We further note that

$$D_2 D_3 U_2 U_1 = \frac{1}{4} D_2 U_1 + \frac{3}{4} D_2 M_2^+ U_1 = \frac{1}{4} D_2 U_1 + \frac{3}{4} D_2 U_1 D_2 U_1 \pm \frac{3}{4} \gamma I \ , \quad D_2 U_1 = \frac{1}{3} I + \frac{2}{3} M_1^+.$$

Plugging in these bounds into (17.1) we get that

$$\begin{aligned}
\frac{1}{6} S &= \frac{1}{2} D_2 U_1 + \frac{1}{3} M_1^+ - D_2 D_3 U_2 U_1 \\
&\preceq \frac{1}{4} D_2 U_1 + \frac{1}{3} M_1^+ - \frac{3}{4} D_2 U_1 D_2 U_1 + \frac{3}{4} \gamma I \\
&= \frac{1}{3} M_1^+ - \frac{1}{2} D_2 U_1 M_1^+ + \frac{3}{4} \gamma I \\
&= M_1^+ \left( \frac{1}{3} I - \frac{1}{2} D_2 U_1 \right) + \frac{3}{4} \gamma I
\end{aligned}$$

By $\gamma$-expansion of $X$, $|\lambda|_2 (D_2 U_1) \leq \frac{2}{3} + 2\gamma$. Therefore we have that $\lambda_2(S) \leq 12\gamma$. A similar argument gives that $\lambda_{\min}(S) \geq -12\gamma$. Therefore $|\lambda|_2 (S) \leq 12\gamma$

As a consequence,

$$\sigma^2 = |\lambda|_2 (B) \leq 12\gamma + O(\gamma^2).$$

□

The other lemma needed in step 2 is the following.

**Lemma 17.9.** *Consider an $X$ that satisfies the conditions in the theorem. For any $v \in X(0)$ such that $\Pr_{t \ni v}[\delta_1 f(t) \neq 0] < \frac{1}{5}$,*

$$\Pr_{e \supset v}[ \Pr_{t \supset e}[\delta_1 f(t) \neq 0] \geq \frac{1}{4} ] \leq \varepsilon \cdot O(\gamma) \ .$$

This lemma will be proved in the next lecture.

Now we finish step 2 using the two lemmas.

Let $T^* = \{t \in X(2) \mid \delta_1 f(t) \neq 0\}$. We shall prove that $wt_2(T^*) \geq \eta$ (where $\eta = \frac{1}{12} - O(\gamma)$) by finding a proper operator $P : C^2 \to C^2$ such that

1. $\vec{1}$ is an eigenvector of $P$ whose corresponding eigenvalue is at most 1, and
2. $\left\langle \vec{1}_{T^*}, P\vec{1}_{T^*} \right\rangle_{\pi(2)} \geq (\eta + \|P\|_{op}) \cdot wt_2(T^*)$.

If such $P$ exists then by spectral decomposition

$$(\eta + |\lambda|_2(P)) \cdot wt_2(T^*) \leq \left\langle \vec{1}_{T^*}, P\vec{1}_{T^*} \right\rangle_{\pi(2)} \leq wt_2(T^*)^2 + |\lambda|_2(P) \cdot wt_2(T^*).$$

Dividing both sides by $wt_2(T^*)$ we get $wt_2(T^*) \geq \eta$. Then we can conclude that $\|\delta_1 f_0\| \geq \eta$ and complete step 2.

We next prove that $P = M_2^+ - U_1 D_2$ satisfies the conditions. First of all $\vec{1}$ is an eigenvector of both $M_2^+$ and $U_1 D_2$. Secondly, by $\gamma$-expansion of $X$, $\left\|M_2^+ - U_1 D_2\right\|_{op} \leq \gamma$. Therefore condition 1 is immediately satisfied, and we move to check condition 2.

First since $\vec{1}_{T^*} = \delta_1 f$,

$$\left\langle \vec{1}_{T^*}, M_2^+\vec{1}_{T^*} \right\rangle_{\pi(2)} \geq \frac{1}{3} \cdot wt_2(T^*).$$

So it suffice to prove that

$$\left\langle \vec{1}_{T^*}, UD\vec{1}_{T^*} \right\rangle_{\pi(2)} \leq \left(\frac{1}{3} - \gamma - \eta\right) \cdot wt_2(T^*).$$

For every edge $ab \in X(1)$, we define the local function $f|_{ab} : X_{ab}(0) \to \mathbb{F}_2$ such that $f|_{ab}(c) = f(abc)$. With this notation we can rewrite the inner product as follows.

$$\begin{aligned}
\left\langle \vec{1}_{T^*}, UD\vec{1}_{T^*} \right\rangle &= \mathbb{E}_{ab\sim\pi(1)}[\Pr_{c,d\sim\pi_{ab}(0)}[abc, abd \in T^*]] \\
&= \mathbb{E}_{a\sim\pi(0), b\sim\pi_a(0)}[\mathbb{E}_{c,d\sim\pi_{ab}(0)}[\delta_1 f|_{ab}(c) \cdot \delta_1 f|_{ab}(d)]] \\
&\leq \Pr_{a\sim\pi(0)}\left[\Pr_{t\ni a}[\delta_1 f(t) \neq 0] \geq \frac{1}{3}\right] + \Pr_{a\sim\pi(0), b\sim\pi_a(0)}\left[\Pr_{t\ni a}[\delta_1 f(t) \neq 0] < \frac{1}{5} \text{ but } \Pr_{t\supset ab}[\delta_1 f(t) \neq 0] \geq \frac{1}{4}\right] \\
&\quad + \mathbb{E}_{ab\sim\pi(1)}\left[\mathbf{1}[\Pr_{t\supset ab}[\delta_1 f(t) \neq 0] < \frac{1}{4}] \cdot \langle \delta_0 f|_{ab}, U_{-1}D_0\delta_0 f|_{ab}\rangle_{\pi_{ab}(1)}\right].
\end{aligned}$$

The first and the second terms in the summand are all of order $o(1) \cdot wt_2(T^*)$, so we just need the last term to be small as well.

Notice that since $X_{ab}$ is a $\gamma$-expander $\left\|M_{ab,0}^+ - U_{-1}D_0\right\|_{op} \leq \gamma$ where $M_{ab,0}^+$ is the non-lazy random walk over $X_{ab}$ which is also a $\gamma$-expander. Therefore $|\lambda|_2(U_{-1}D_0) \leq 2\gamma$. As a result, for every edge $ab$ that contributes to the third summand

$$\langle \delta_0 f|_{ab}, U_{-1}D_0\delta_0 f|_{ab}\rangle_{\pi_{ab}(1)} \leq \|\delta_0 f|_{ab}\|^2 + 2\gamma\|\delta_0 f|_{ab}\| \leq (1/4 + 2\gamma)\|\delta_0 f|_{ab}\|.$$

Now taking expectation over the good $ab$'s we see that the last term is bounded by

$$\begin{aligned}
\mathbb{E}_{ab\sim\pi(1)}\left[\mathbf{1}[\Pr_{t\supset ab}[\delta_1 f(t)\neq 0]<\frac{1}{4}]\cdot\langle\delta_0 f|_{ab},U_{-1}D_0\delta_0 f|_{ab}\rangle_{\pi_{ab}(1)}\right] &\leq (1/4+2\gamma)\cdot\mathbb{E}_{ab\sim\pi(1)}\left[\|\delta_0 f|_{ab}\|\right]\\
&=(1/4+2\gamma)\cdot\mathbb{E}_{ab\sim\pi(1)}\left[\Pr_{t\supset ab}[\delta_1 f(t)\neq 0]\right]\\
&=(1/4+2\gamma)\cdot wt_2(T^*)
\end{aligned}$$

So combined together we get that

$$\left\langle\vec{1}_{T^*},UD\vec{1}_{T^*}\right\rangle_{\pi(2)}<\left(\frac{1}{4}+O\left(\frac{\gamma}{\beta^2}\right)\right)\cdot wt_2(T^*)\leq\left(\frac{1}{3}-\gamma-\eta\right)\cdot wt_2(T^*).$$

Thereby we complete step 2.

**590.05 Expanders and High-dimensional Expanders** **Fall 2025**

# Lecture 19: Locally Testable Codes

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

## 19.1 The left-right Cayley complex and code

**Definition 19.1** (Left-right Cayley Complex)**.** *Consider a group $G$ and two sets of generators $A = A^{-1}$ and $B = B^{-1}$ such that $ag \neq gb$ for all $g \in G, a \in A, b \in B$ (to ensure regularity of the final complex). We define the left-right Cayley complex $X$ (with respect to $G, A, B$) as:*

$$X(0) = G^{(0)} \sqcup G^{(1)}, \quad X(1) = \{g^{(0)}, ag^{(1)}\}_{g \in G, a \in A} \cup \{g^{(0)}, gb^{(1)}\}_{g \in G, b \in B},$$

$$X(2) = \{\{g^{(0)}, ag^{(1)}, agb^{(0)}, gb^{(1)}\}\}_{g \in G, a \in A, b \in B}.$$

For ease of notation we write $X(1) = X_A(1) \bigsqcup X_B(1)$ where $X_A(1)$ are edges connected via elements in $A$ and $X_B(1)$ are edges connected via elements in $B$.

We note that we can analogously define links of vertices and edges in $X$. We use

$$X_g(0) = \{e \in X(1) \mid g \in e\}, \; X_g(1) = \{s \in X(2) \mid g \in s\},$$

and

$$X_e(0) = \{s \in X(2) \mid e \subset s\}.$$

We note that the vertex links $X_g$ are complete bipartite graphs, and for any $\{g, g'\} \in X(1)$

$$X_g(1) \cap X_{g'}(1) = X_{\{g,g'\}}(0).$$

**Definition 19.2** (Codes on $X$)**.** *Given two local linear codes $C_A \subset \mathbb{F}^A$ and $C_B \subset \mathbb{F}^B$, we can defined the code on $X$ as*

$$C(X) = \{f \mid \forall e \in X_A(1), \; f|_{X_e(0)} \in C_B \text{ and } \forall e \in X_B(1), \; f|_{X_e(0)} \in C_A\}.$$

Here we show two results about the rate and distance of $C(X)$.

**Claim 19.3.** *Use $r_A, r_B$ to denote the rate of $C_A$ and $C_B$. If $r_A, r_B > \frac{1}{2}$, then the rate of $C(X)$ is $\Theta(1)$.*

*Proof.* Let $|G|, |A|, |B|$ denote the cardinalities of $G, A, B$ respectively. Each face in $X(2)$ corresponds to a triple $(g, a, b) \in G \times A \times B$, so

$$N = |X(2)| = |G| \cdot |A| \cdot |B|$$

is the blocklength of the global code $C(X)$.

Each $A$-edge $e \in X_A(1)$ is incident to $|B|$ faces. The local constraint $f|_{X_e(0)} \in C_B$ thus imposes $(1 - r_B)|B|$ linear constraints on the global coordinates. Similarly, each $B$-edge $e \in X_B(1)$ is incident to $|A|$ faces, and the condition $f|_{X_e(0)} \in C_A$ contributes $(1 - r_A)|A|$ linear constraints .

Since $|X_A(1)| = |G| \cdot |A|$ and $|X_B(1)| = |G| \cdot |B|$, the total number of linear constraints is at most

$$|G| \cdot |A| \cdot (1 - r_B)|B| + |G| \cdot |B| \cdot (1 - r_A)|A| = |G| \cdot |A| \cdot |B| \cdot (2 - r_A - r_B).$$

Hence

$$\dim(C(X)) \;\geq\; N - |G|\cdot|A|\cdot|B|\cdot(2 - r_A - r_B) = |G|\cdot|A|\cdot|B|(r_A + r_B - 1).$$

Dividing by $N = |G|\cdot|A|\cdot|B|$ gives the lower bound

$$\text{rate}(C(X)) \;\geq\; r_A + r_B - 1.$$

If $r_A, r_B > \frac{1}{2}$ then $r_A + r_B - 1 > 0$, so the global rate is bounded below by a positive constant. In particular, the rate of $C(X)$ is $\Theta(1)$. □

**Claim 19.4.** *Use $\delta_A, \delta_B$ to denote the relative distance of $C_A$ and $C_B$. If the Cayley graphs $Cay(G, A)$ and $Cay(G, B)$ are both two-sided $\lambda$-expanders where $\lambda < \min(\delta_A, \delta_B)$, then the relative distance of $C(X)$ is $\Theta(1)$.*

*Proof.* Let $f \in C(X)$ be a nonzero codeword and let $S = \text{supp}(f) \subseteq X(2)$ be its support (the set of faces on which $f$ is nonzero).

Define the *active vertex set*

$$V^* := \{v \in X(0) \mid f|_{X_v(1)} \not\equiv 0\},$$

i.e. the set of vertices incident to at least one face in $S$. We first show that $V^*$ must be a constant fraction of $V$; then we derive a constant lower bound on $|S|$ from the fact that each active vertex forces many incident faces to lie in $S$.

The 1-skeleton $X$ is the $d$-regular bipartite graph where $d = |A| + |B|$.

Since both $\text{Cay}(G, A)$ and $\text{Cay}(G, B)$ are two-sided $\lambda$-expanders, their union's double cover $X^{\leq 1}$ is a one-sided $\lambda$-expander.

Let $E^* \subseteq X(1)$ be the set of edges $e \in X(1)$ for which the local constraint seen at $e$ is nonzero, i.e.

$$E^* := \{e \in X(1) : \; f|_{X_e(0)} \not\equiv 0\}.$$

Every active vertex $v \in V^*$ is incident to at least $(\delta_A\cdot|A| + \delta_B\cdot|B|)$ edges in $E^*$, hence at least $(\delta_A\cdot|A| + \delta_B\cdot|B|)$ neighbors of $v$ are also in $V^*$. Using spectral decomposition of $X^{\leq 1}$'s normalized adjacency matrix $P$ we get that

$$\min(\delta_A, \delta_B)\cdot wt_0(V^*) \leq \left\langle \vec{1}_{V^*}, P\vec{1}_{V^*} \right\rangle \leq wt_0(V^*)^2 + \lambda wt_0(V^*).$$

Hence the weight of $V^*$ is at least

$$wt_0(V^*) \geq \min(\delta_A, \delta_B) - \lambda = \Theta(1).$$

Each vertex $v \in V^*$ is incident to at least $\delta_A|A|\cdot\delta_B|B|$ faces in $S$. A single face is incident to exactly 4 vertices, so summing the per-vertex counts over $V^*$ counts each face at most 4 times. Consequently

$$|S| \;\geq\; \frac{1}{4}\sum_{v\in V^*} \delta_A|A|\cdot\delta_B|B| \geq \frac{1}{4}\cdot\delta_A|A|\cdot\delta_B|B|\cdot\Theta(|G|) = \Theta(1)|A|\cdot|B|\cdot|G| = \Theta(1)\cdot|X(2)|\,.$$

Hence every nonzero codeword has relative weight at least $\Theta(1)$. This proves the claim. □

## 19.2 Local testability of $C(X)$

Recall that we say a code test pair $(C, T)$ is locally testable if for every $f$

$$\text{dist}(f, C) \le \Pr_T[T \text{ rejects } f].$$

We now define a natural local test for $C(X)$:

1. Sample a random $g \in X(0)$,
2. Check if $f|_{X_g(1)} \in C_A \otimes C_B$.

To analyze this test, we introduce the notion of $\beta$-*agreement testability* for tensor codes.

**Definition 19.5** ($\beta$-agreement testability)**.** *An infinite family of tensor codes*

$$\{\, C_A \otimes C_B \subseteq \mathbb{F}^{\Delta\times\Delta} \,\}_{\Delta\in\mathbb{N}}$$

*is said to be* $\beta$-agreement testable *if, for every collection of local strings* $\{w_x \in \mathbb{F}^{\Delta}\}_{x\in A\cup B}$*, there exists a global codeword* $c \in C_A \otimes C_B$ *such that*

$$\beta \cdot \left( \frac{1}{2} \Pr_{a\in A}[c(a,\cdot) \ne w_a] + \frac{1}{2} \Pr_{b\in B}[c(\cdot,b) \ne w_b] \right) \le \Pr_{a,b\sim A\times B}[w_a(b) \ne w_b(a)].$$

For simplicity, we assume throughout that $|A| = |B| = \Delta$.

**Theorem 19.6.** *Suppose the Cayley graphs* $\text{Cay}(G, A)$ *and* $\text{Cay}(G, B)$ *are both two-sided* $\lambda$*-expanders with* $\lambda = o_\Delta(1)$*, and that* $C_A \otimes C_B$ *is* $\beta$*-agreement testable for some* $\beta \in \Theta_\Delta(1)$*. Then the pair* $(C(X), T)$ *is an* $\eta$*-locally testable code with*

$$\eta = \Theta(\Delta^{-1}).$$

*That is, for every* $f \in \mathbb{F}^{X(2)}$*, there exists* $c \in C(X)$ *such that*

$$\eta \cdot \text{dist}(f, c) \le \Pr_{g\in X(0)}[f|_{X_g(1)} \notin C_A \otimes C_B].$$

For brevity, denote

$$\varepsilon := \Pr_{g\in X(0)}[f|_{X_g(1)} \notin C_A \otimes C_B].$$

The proof follows the structure of the proof of the local-to-global theorem for cosystolic expanders. Given an input word $f_0$, our goal is to either (i) find a nearby codeword $c \in C(X)$, or (ii) show that $\Pr_{g\in X(0)}[f|_{X_g(1)} \notin C_A \otimes C_B] \ge \eta$.

## Iterative Correction Procedure

Given $f_0$, we iteratively construct a consistent collection of local codewords:

1. For each $g \in X(0)$, choose $c_g \in C_A \otimes C_B$ minimizing $\text{dist}(c_g, f_0|_{X_g(1)})$. Initialize $f := f_0$.
2. Define the current set of disagreements:
$$E' = \left\{ \{g, g'\} \in X(1) \;\middle|\; c_g|_{X_{\{g,g'\}}(0)} \ne c_{g'}|_{X_{\{g,g'\}}(0)} \right\}.$$
3. If there exists $g \in X(0)$ and an alternative local codeword $c'_g \in C_A \otimes C_B$ that decreases $|E'|$, update
$$c_g := c'_g \quad \text{and} \quad f|_{X_g(1)} := c'_g.$$

4. Repeat steps 2–3 until no such improvement exists. If at termination $E' = \emptyset$, output the consistent codeword $c$ induced by the $\{c_g\}$. Otherwise, output the current $f$.

Define the following sets:

$V_0 = \{\, g \mid \text{in step 1 } c_g \neq f_0|_{X_g(1)} \,\}$, $E'_0$ = initial disagreement set $E'$, $V' = \{\, g \mid c_g \text{ was updated at least once} \,\}$.

## Case 1: The procedure outputs a codeword

If the final output is a consistent codeword $c$, then

$$\mathrm{dist}(f_0, c) \leq wt_0(V_0) + wt_0(V') = \varepsilon + wt_0(V') \leq \varepsilon + \frac{|E'_0|}{|X(0)|}.$$

Since each vertex has degree $\Delta$,

$$\frac{|E'_0|}{|X(0)|} = \Delta \cdot wt_1(E'_0).$$

Furthermore,

$$\begin{aligned} wt_1(E'_0) &= \Pr_{(g,g')\in X(1)}[\, c_g|_{X_{\{g,g'\}}(0)} \neq c_{g'}|_{X_{\{g,g'\}}(0)} \,] \\ &\leq \Pr_{(g,g')\in X(1)}\big[ f|_{X_g(1)} \notin C_A \otimes C_B \text{ or } f|_{X_{g'}(1)} \notin C_A \otimes C_B \big] \\ &\leq 2 \Pr_{g\in X(0)}[f|_{X_g(1)} \notin C_A \otimes C_B] = 2\varepsilon. \end{aligned}$$

Thus

$$(2\Delta + 1)^{-1} \cdot \mathrm{dist}(f_0, c) \leq \varepsilon.$$

## Case 2: The procedure terminates with $f \notin C(X)$

If the final $f$ is not globally consistent, then in the final step the disagreement set $E'$ is nonempty. We will show that

$$2\eta \leq wt_1(E') \leq wt_1(E'_0) \leq 2\varepsilon.$$

To do so, consider the random-walk operators $U_0 M_0^+ D_1$ and $M_\|$ acting on edges of $X(1)$:

- In $U_0 M_0^+ D_1$, we start at an edge $e$, pick a random vertex $g \in e$, move to a random neighbor $g'$ of $g$ in $X^{\leq 1}$, and finally to a random edge $e' \ni g'$.
- In $M_\|$, we start from an edge $\{g, g'\}$. If $\{g, g'\} \in X_A(1)$, move to $\{gb, g'b\}$ for random $b \in B$; if $\{g, g'\} \in X_B(1)$, move to $\{ag, ag'\}$ for random $a \in A$.

Observe that $\lambda_2(M_0^+) \leq \lambda = o_\Delta(1)$, and that $M_\|$ decomposes into a disjoint union of $2\Delta$ graphs

$$\bigcup_{a\in A} G_a \;\cup\; \bigcup_{b\in B} G_b,$$

where

$$V(G_a) = \{\{g^{(0)}, ag^{(1)}\}\}_{g^{(0)}\in G^{(0)}}, \quad E(G_a) = \{\{e, e'\} \mid \exists b \in B,\ e' = eb\},$$

and $G_b$ is defined analogously. Each $G_a$ is isomorphic to $\mathrm{Cay}(G, B)$ and each $G_b$ to $\mathrm{Cay}(G, A)$. Therefore $M_\|$ is a disjoint union of $2\Delta$ two-sided $\lambda$-expanders.

We will complete the proof in the next lecture.

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# Lecture 21: Locally Testable Codes (cont.)

*Instructor: Siqi Liu* — *Scribe: Siqi Liu*

For any edge $(g^{(0)}, ag^{(1)}) \in X(1)$, partition all edges that share a square with it into the following three sets:

$$E_0 = \{(g^{(0)}, gb^{(1)}) \mid b \in B\}, \quad E_1 = \{(ag^{(1)}, agb^{(0)}) \mid b \in B\}, \quad E_{\|} = \{(gb^{(1)}, agb^{(0)}) \mid b \in B\}.$$

Note that an analogous partition exists for every $(g^{(0)}, gb^{(1)})$, but we omit it here for simplicity. For brevity, we also drop the superscripts in the following discussion.

**Claim 21.1.** *If $(g, ag) \in E'$, then*

$$|E_0 \cup E_1 \cup E_{\|} \cap E'| \geq \delta_B \cdot \Delta.$$

*Proof.* Since $(g, ag) \in E'$, we have $c_g|_{X_{(g,ag)}(1)}, c_{ag}|_{X_{(g,ag)}(1)} \in C_B$, but $c_g \neq c_{ag}$. Hence, the two codewords differ on at least $\delta_B \cdot \Delta$ squares.

For each square $\sigma = (g, ag, agb.gb) \in X_{(g,ag)}(0)$ where $c_g(\sigma) \neq c_{ag}(\sigma)$, we must have

$$c_g(\sigma) \neq c_{gb}(\sigma) \vee c_{gb}(\sigma) \neq c_{agb}(\sigma) \vee c_{agb}(\sigma) \neq c_{ag}(\sigma).$$

This implies that at least one of the other edges in $\sigma$ lies in $E'$.

Therefore,

$$|E_0 \cup E_1 \cup E_{\|} \cap E'| \geq \delta_B \cdot \Delta.$$

□

Let $\{c_g\}_{g \in X(0)}$ denote the collection of local codes at the end of the correction procedure. The $\beta$-agreement testability of the tensor code relates local disagreement probabilities as follows:

**Claim 21.2.** *For every $g \in X(0)$,*

$$\beta \cdot \Pr_{(g,g') \sim X_g(0)}[(g, g') \in E'] \leq \Pr_{a,b \sim A \times B}[(ag, agb) \text{ or } (gb, agb) \in E'].$$

*Proof.* For every $a \in A$, $b \in B$, let $w_a = c_{ag}|_{X_{(g,ag)}(0)}$ and $w_b = c_{gb}|_{X_{(g,gb)}(0)}$. Then, By the $\beta$-agreement testability of $C_A \otimes C_B$, there exists a codeword $c$ such that:

$$\beta \cdot \left( \frac{1}{2} \Pr_{a \in A}[c(a, \cdot) \neq w_a] + \frac{1}{2} \Pr_{b \in B}[c(\cdot, b) \neq w_b] \right) \leq \Pr_{a,b \sim A \times B}[w_a(b) \neq w_b(a)].$$

We rewrite the LHS as:

$$\begin{aligned} \beta \cdot \Pr_{(g,g') \sim X_g(0)}[c|_{X_{(g,g')}(0)} \neq c_{g'}|_{X_{(g,g')}(0)}] &\geq \min_{c_g \in C_A \otimes C_B} \beta \cdot \Pr_{(g,g') \sim X_g(0)}[c_g|_{X_{(g,g')}(0)} \neq c_{g'}|_{X_{(g,g')}(0)}] \\ &= \beta \cdot \Pr_{(g,g') \sim X_g(0)}[(g, g') \in E']. \end{aligned}$$

For the RHS:

$$\begin{aligned}\Pr_{a,b\sim A\times B}[c_{ag}(g,ag,agb,gb)\neq c_{gb}(g,ag,agb,gb)] &\leq \Pr_{a,b\sim A\times B}[c_{ag}|_{X_{(ag,agb)}}\not\equiv c_{agb}|_{X_{(ag,agb)}} \\ &\qquad \vee\ c_{gb}|_{X_{(gb,agb)}}\not\equiv c_{agb}|_{X_{(gb,agb)}}] \\ &= \Pr_{a,b\sim A\times B}[(ag,agb) \text{ or } (gb,agb)\in E'].\end{aligned}$$

Combining both sides yields the claim. □

Now consider the random walk on $X(1)$ with transition matrix $M=\alpha M_{\|}+(1-\alpha)U_0M_0^+D_1$.

Given $(g,ag)\in E'$, define the following quantities:

$$r_0=|E_0\cap E'|,\quad r_1=|E_1\cap E'|,\quad r_{\|}=|E_{\|}\cap E'|.$$

We prove the following bound on the probability that a random neighbor of a corrupted edge remains corrupted:

**Claim 21.3.** *Given an edge $e\in E'$, the probability that a random neighbor $e'$ of $e$ in the walk defined by $M$ also lies in $E'$ satisfies:*

$$\Pr_{e'}[e'\in E']\geq \alpha\cdot\frac{r_{\|}}{\Delta}+(1-\alpha)\cdot\frac{\beta}{16}\cdot\frac{r_0+r_1}{\Delta}.$$

*Proof.* With probability $\alpha$, the walk moves from $e$ to a random edge in $E_{\|}$, which has size $\Delta$. In this case, the probability that $e'\in E'$ is exactly $\frac{r_{\|}}{\Delta}$.

Otherwise, with probability $1-\alpha$, the walk follows the path defined by $U_0M_0^+D_1$. This step involves eight possible types of transitions based on the types of edges in $M_0^+$ and $U_0$:

$$(g,ag)\xrightarrow{D_1} g_1=\begin{cases} g \\ ag\end{cases}\xrightarrow{M_0^+}\begin{cases} a_1g \text{ or } a_1ag \\ gb_1 \text{ or } agb_1\end{cases}\xrightarrow{U_0}\begin{cases}(a_1g,a_2a_1g),\ (a_1g,a_1gb_2)\\ (gb_1,a_2gb_1),\ (gb_1,gb_1b_2)\\ (a_1ag,a_2a_1ag),\ (a_1ag,a_1agb_2)\\ (agb_1,a_2agb_1),\ (agb_1,agb_1b_2)\end{cases}$$

Now consider the cases where the $M_0^+$ and $U_0$ transitions use different edge types. For instance, we have:

$$(a_1g,a_1gb_2),\ (gb_1,a_2gb_1),\quad \text{for random } a_1,a_2\in A,\ b_1,b_2\in B.$$

We aim to lower bound the probability:

$$\Pr_{a_1\in A,b_2\in B}[(a_1g,a_1gb_2)\in E']+\Pr_{a_2\in A,b_1\in B}[(gb_1,a_2gb_1)\in E'].$$

By Claim 21.3,

$$\begin{aligned}\Pr_{a_1\in A,b_2\in B}[(a_1g,a_1gb_2)\in E']+\Pr_{a_2\in A,b_1\in B}[(gb_1,a_2gb_1)\in E'] &\geq \Pr_{a',b'\in A\times B}[(a'g,a'gb') \text{ or } (gb',a'gb')\in E'] \\ &\geq \beta\cdot\Pr_{(g,g')\sim X_g(0)}[(g,g')\in E'] \\ &\geq \frac{\beta}{2}\cdot\Pr_{b\in B}[(g,gb)\in E']=\frac{\beta}{2}\cdot\frac{r_0}{\Delta}.\end{aligned}$$

A symmetric argument gives:

$$\Pr_{a_1 \in A, b_2 \in B}[(a_1ag, a_1agb_2) \in E'] + \Pr_{a_2 \in A, b_1 \in B}[(agb_1, a_2agb_1) \in E'] \geq \frac{\beta}{2} \cdot \frac{r_1}{\Delta}.$$

Therefore, for a random neighbor $e'$ of $e$ in $U_0 M_0^+ D_1$:

$$\Pr_{e'}[e' \in E'] \geq \frac{1}{8} \cdot \frac{\beta}{2} \cdot \frac{r_0 + r_1}{\Delta} = \frac{\beta}{16} \cdot \frac{r_0 + r_1}{\Delta}.$$

Combining both bounds completes the proof. □

Now, setting $\alpha = \frac{\beta}{16+\beta}$, we get:

$$\Pr_{e'}[e' \in E'] \geq \frac{\beta}{16 + \beta} \cdot \frac{r_\| + r_0 + r_1}{\Delta} \geq \frac{\beta \delta_B}{16 + \beta}.$$

If $e = (g, gb)$, a similar bound holds with $\delta_B$ replaced by $\delta_A$. Thus,

$$\langle \vec{1}_{E'}, M \vec{1}_{E'} \rangle \geq \mathrm{wt}_1(E') \cdot \frac{\beta \cdot \min(\delta_A, \delta_B)}{16 + \beta}.$$

Hence, either

$$\langle \vec{1}_{E'}, M_\| \vec{1}_{E'} \rangle \geq \mathrm{wt}_1(E') \cdot \frac{\beta \cdot \min(\delta_A, \delta_B)}{16 + \beta},$$

or

$$\langle \vec{1}_{E'}, U_0 M_0^+ D_1 \vec{1}_{E'} \rangle \geq \mathrm{wt}_1(E') \cdot \frac{\beta \cdot \min(\delta_A, \delta_B)}{16 + \beta}.$$

If the first inequality holds, then there exists a connected component $G_x$ in $M_\|$ such that

$$\langle \vec{1}_{E'}, M_\||_{G_x} \vec{1}_{E'} \rangle_{G_x} \geq \mathrm{wt}_{G_x}(E') \cdot \frac{\beta \cdot \min(\delta_A, \delta_B)}{16 + \beta} \overset{\text{by expansion of } G_x}{\Longrightarrow} \mathrm{wt}_{G_x}(E') \geq \frac{\beta \cdot \min(\delta_A, \delta_B)}{16 + \beta} - \lambda = \Theta(1).$$

So $\mathrm{wt}_1(E') \geq \Theta(\Delta^{-1})$.

If the second inequality holds, then

$$\langle \vec{1}_{E'}, U_0 M_0^+ D_1 \vec{1}_{E'} \rangle \geq \mathrm{wt}_1(E') \cdot \frac{\beta \cdot \min(\delta_A, \delta_B)}{16 + \beta} \overset{\text{by expansion of } U_0 M_0^+ D_1}{\Longrightarrow} \mathrm{wt}_1(E') \geq \Theta(1).$$

Thus, in either case, we conclude:

$$\Pr_{g \in X(0)}[f_0|_{X_g(1)} \notin C_A \otimes C_B] \geq \frac{\mathrm{wt}_1(E'_0)}{2} \geq \frac{\mathrm{wt}_1(E')}{2} = \Theta_\Delta(1).$$

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# Lecture 22: Quantum low density parity check codes

*Instructor: Siqi Liu* *Scribe: Siqi Liu*

In this lecture, we delve into a construction of quantum low-density parity-check codes (qLDPCs) with linear distance using the left-right Cayley complex introduced in previous lectures.

## 22.1 Calderbank–Shor–Steane (CSS) codes

CSS codes are a class of quantum codes constructed from pairs of classical codes.

Let $C_X, C_Z \subset \mathbb{F}_2^n$ be two classical codes satisfying the orthogonality condition $C_X^\perp \subset C_Z$ (equivalently, $C_Z^\perp \subset C_X$). Then the associated CSS code is defined as:

$$\mathrm{CSS}(C_X, C_Z) = \{f + C_X^\perp \mid f \in C_Z\}.$$

In other words, a codeword corresponds to a coset of $C_X^\perp$ within $C_Z$.

The rate and relative distance of the CSS code are given by:

$$r(\mathrm{CSS}(C_X, C_Z)) = \frac{\dim(C_Z) - \dim(C_X^\perp)}{n} = \frac{\dim(C_X) + \dim(C_Z)}{n} - 1,$$

$$\delta(\mathrm{CSS}(C_X, C_Z)) = \min\left(\min_{f \in C_Z \setminus C_X^\perp} \|f\|,\ \min_{f \in C_X \setminus C_Z^\perp} \|f\|\right).$$

We now construct such a code using the left-right Cayley complex that achieves constant rate and relative distance.

## 22.2 CSS code from the left-right Cayley complex

Given a left-right Cayley complex $X$ with $|A| = |B| = \Delta$, we define $C_X$ and $C_Z$ as follows:

$$C_X = \{f \in \mathbb{F}^{X(2)} \mid \forall g \in G^0,\ f|_{X_g(1)} \in (C_A \otimes C_B)^\perp\},$$

$$C_Z = \{f \in \mathbb{F}^{X(2)} \mid \forall g \in G^1,\ f|_{X_g(1)} \in (C_A^\perp \otimes C_B^\perp)^\perp\}.$$

**Claim 22.1.** *The pair $(C_X, C_Z)$ defines a valid CSS code: $C_X^\perp \subset C_Z$.*

*Proof.* Let $f \in C_X^\perp$. Then for all $c \in C_X$, we have $\langle f, c\rangle = 0$. Since $c|_{X_g(1)} \in (C_A \otimes C_B)^\perp$ for all $g \in G^0$, it follows that $f|_{X_g(1)} \in C_A \otimes C_B$. Moreover, $(C_A \otimes C_B) \subset (C_A^\perp \otimes C_B^\perp)^\perp$, so we have $f \in C_Z$. Therefore, $C_X^\perp \subset C_Z$. □

**Claim 22.2.**

$$r(\mathrm{CSS}(C_X, C_Z)) \geq r_A + r_B - 2r_A r_B.$$

*Proof.* Let $n = |X(2)|$. Observe that $C_X$ imposes local constraints for each $g \in G^0$, and $C_Z$ imposes similar constraints for each $g \in G^1$.

For each vertex $g \in G^0$, the local constraint corresponds to the orthogonal code $(C_A \otimes C_B)^\perp$, which has dimension

$$\dim((C_A \otimes C_B)^\perp) = \Delta^2 - \dim(C_A \otimes C_B) = \Delta^2(1 - r_A r_B).$$

Hence,

$$\dim(C_X) \geq n - |G^0|\Delta^2(1 - r_A r_B).$$

Similarly, for each $g \in G^1$, we have

$$\dim(C_Z) \geq n - |G^1|\Delta^2(1 - (1 - r_A)(1 - r_B)) = n - |G^1|\Delta^2(r_A + r_B - r_A r_B).$$

Since $|G^0| = |G^1| = |G|$, we get:

$$\begin{aligned}\dim(C_X) + \dim(C_Z) &\geq 2n - |G|\Delta^2\big[(1 - r_A r_B) + (r_A + r_B - r_A r_B)\big] \\ &= 2n - |G|\Delta^2(1 - r_A - r_B + 2r_A r_B).\end{aligned}$$

Therefore, using

$$r(\mathrm{CSS}(C_X, C_Z)) = \frac{\dim(C_X) + \dim(C_Z)}{n} - 1,$$

we obtain

$$r(\mathrm{CSS}(C_X, C_Z)) = r_A + r_B - 2r_A r_B.$$

□

### Robust locally testable tensor codes

**Definition 22.3** ($\omega$-robust locally testable tensor codes)**.** *A tensor code $C_A \otimes C_B \subset \mathbb{F}^{\Delta\times\Delta}$ with $\delta = \min(\delta(C_A), \delta(C_B))$ is $\omega$-robust locally testable if:*

1. *For all $f \in C_A \otimes \mathbb{F}^\Delta \oplus \mathbb{F}^\Delta \otimes C_B$ with $\|f\| \leq \omega$, there exist $r \in C_A \otimes \mathbb{F}^\Delta$ and $c \in \mathbb{F}^\Delta \otimes C_B$, each supported on at most $\omega\Delta/\delta$ rows (for $r$) or columns (for $c$), such that:*

$$f = r + c.$$

2. *There exists $\eta > 0$ independent of $\Delta$ such that for all $f \in \mathbb{F}^{\Delta\times\Delta}$,*

$$\mathrm{dist}(f, C_A \otimes C_B) \leq \frac{1}{2\eta}\left(\mathrm{dist}(f, C_A \otimes \mathbb{F}^\Delta) + \mathrm{dist}(f, \mathbb{F}^\Delta \otimes C_B)\right).$$

## 22.3 Main theorem

**Theorem 22.4.** *If the Cayley graphs* $\mathrm{Cay}(G, A)$ *and* $\mathrm{Cay}(G, B)$ *are Ramanujan, the linear codes $C_A, C_B, C_A^\perp, C_B^\perp \subseteq \mathbb{F}_2^\Delta$ all have relative distance at least $\delta > 0$, and both $C_A \otimes C_B$ and $C_A^\perp \otimes C_B^\perp$ are $\frac{1}{\Delta^{\frac{1}{2}-\varepsilon}}$-robust locally testable for some $\varepsilon \in (0, \frac{1}{2})$, then*

$$\delta(\mathrm{CSS}(C_X, C_Z)) \geq \frac{\delta}{4\Delta^{\frac{3}{2}+\varepsilon}}.$$

We will prove that $d_Z \geq \frac{\delta}{4\Delta^{\frac{3}{2}+\varepsilon}}$. A similar argument establishes the bound for $d_X$, which we leave as an exercise.

The main idea is: given $f \in C_Z$ with $\|f\| < \frac{\delta}{4\Delta^{\frac{3}{2}+\varepsilon}}$, we construct local corrections $c_g \in C_X^{\perp}$ supported on $X_g(1)$ for some $g \in G^0$ such that:

$$\|f + c_g\| < \|f\|.$$

Iterating this process eventually reduces $f$ to the zero vector, implying that $f \in C_X^{\perp}$. Therefore, any such $f$ cannot be a codeword in $C_Z \setminus C_X^{\perp}$, proving the claimed lower bound on $d_Z$.

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# Lecture 23: Quantum low density parity check codes (cont.)

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Let $f \in C_Z$ be an output of the iterative correction procedure such that $f \neq \mathbf{0}$. Define the following vertices sets

$$V_1 = \{g \in G^1 \mid f|_{X_g(1)} \not\equiv \mathbf{0}\},$$

$$V_{\leq} = \{g \in V_1 \mid \left\|f|_{X_g(1)}\right\| \leq \frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}\},\ V_{>} = V_1 \setminus V_{\leq}.$$

**Claim 23.1.** *Let $X$ be a left-right Cayley complex such that $Cay(G, A)$ and $Cay(G, B)$ are Ramanujan with $\Delta \geq 5$. Let $f \in C_Z$ be such that $\|f\| \leq \frac{\delta}{4\Delta^{\frac{3}{2}+\varepsilon}}$. Use $P_s$ to denote the transition matrix of the non-lazy walk on $G^1$ that starts from a $g$ and goes to a random $g'$ that shares a square with $g$. Use $\pi$ to denote the stationary distribution of $P_s$, and $wt$ to denote the weight of a vertex set in $\pi$.*

$$wt(V_>) \leq \frac{49}{\Delta^{1-2\varepsilon}} \cdot wt(V_1).$$

*Proof.* For every $g \in V_1$, $f|_{X_g(1)} \in C_A \otimes \mathbb{F}^\Delta \oplus \mathbb{F}^\Delta \otimes C_B \setminus \{\mathbf{0}\}$. Since the code $C_A \otimes \mathbb{F}^\Delta \oplus \mathbb{F}^\Delta \otimes C_B$ has relative distance at least $\frac{\delta}{\Delta}$, we have that

$$wt(V_1) \cdot \frac{\delta}{\Delta} \leq \cdot \Pr_{g\sim\pi}[\left\|f|_{X_g(1)}\right\|] = \|f\| \leq \frac{\delta}{4\Delta^{\frac{3}{2}+\varepsilon}}.$$

Therefore we have $wt(V_1) \leq \frac{1}{4\Delta^{\frac{1}{2}+\varepsilon}}$.

Note that $\lambda_2(P_s) = \lambda_2(P_A P_B) \leq \frac{4}{\Delta-1} + o(1) \leq \frac{5}{\Delta}$ where $P_A$ and $P_B$ are the random walk operators over $Cay(G, A)$ and $Cay(G, B)$. Therefore spectral decomposition gives that

$$wt(V_>) \cdot \frac{1}{\Delta^{\frac{1}{2}+\varepsilon}} \leq \left\langle \vec{1}_{V_>}, P_s \vec{1}_{V_1} \right\rangle \leq wt(V_>) \cdot wt(V_1) + \lambda^2 \sqrt{wt(V_>) \cdot wt(V_1)}.$$

Combining the lower bound and upper bound, we derive that $wt(V_>) \leq \frac{49}{\Delta^{1-2\varepsilon}} \cdot wt(V_1)$ □

Now consider the 1-skeleton $X^{\leq 1}$ of the left-right Cayley complex. We shall use $M_0^+$ to denote the random walk operator on this graph. We further define sets

$$E_h = \{(g^{(0)}, g^{(1)}) \mid g^{(1)} \in V_{\leq},\ \left\|f|_{X_{(g^{(0)},g^{(1)})}(0)}\right\| \geq \delta - \frac{1}{\delta\Delta^{\frac{1}{2}+\varepsilon}}\},$$

and

$$T = \{g \in G^0 \mid \exists g' \in V_{\leq}\ s.t.\ (g, g') \in E_h\}.$$

In other words $T$ are neighbors of $V_{\leq}$ via some heavy edges in $E_h$. The final $c_g$ will be constructed for some $g \in T$. We now prove the following bound on the weight of $T$ in $M_0^+$.

**Claim 23.2.** *Let $X$ and $f$ satisfy the conditions in Claim 23.1. Use $wt$ to denote the weight of a set in the left-right Cayley complex $X$. Then*

$$wt_0(T) \leq \frac{21}{\delta^2\Delta} \cdot wt_0(V_1).$$

*Proof.* Since the two Cayley graphs $Cay(G,A)$ and $Cay(G,B)$ are Ramanujan, we have that $\lambda_2(M_0^+) \leq \frac{2}{\sqrt{\Delta-1}}$. Since every $g \in T$ is adjacent to a heavy edge in $E_h$, so at least $\Delta \cdot \left(\delta - \frac{1}{\delta\Delta^{\frac{1}{2}+\varepsilon}}\right)$ adjacent edges of $g$ satisfies that $f|_{X_e(0)} \not\equiv \mathbf{0}$. Therefore

$$\left\langle \vec{1}_T, M_0^+ \vec{1}_{V_1} \right\rangle \geq wt_0(T) \cdot \frac{1}{2} \cdot \left(\delta - \frac{1}{\delta\Delta^{\frac{1}{2}+\varepsilon}}\right).$$

Again applying spectral decomposition we have that

$$\left\langle \vec{1}_T, M_0^+ \vec{1}_{V_1} \right\rangle \leq wt_0(T) \cdot wt_0(V_1) + \frac{2}{\sqrt{\Delta-1}}\sqrt{wt_0(T) \cdot wt_0(V_1)}.$$

Combining the lower bound and upper bound together and plugging in $\frac{1}{\Delta^{\frac{1}{2}+\varepsilon}} \ll \delta^2$ to get that $wt_0(T) \leq \frac{21}{\delta^2\Delta} \cdot wt_0(V_1)$. □

Our goal is to find some $g \in T$ such that most of its adjacent edges are in $E_h$. So we use an averaging argument to show that

**Claim 23.3.** *Let $X$ and $f$ satisfy the conditions in Claim 23.1. Then at least $\frac{\delta^2}{168}$-fraction of vertices in $T$ has at least $\frac{\delta^2}{84}$-fraction of the adjacent edges are in $E_h$.*

*Proof.* First, by the $\frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}$-robust locally testability of $C_A \otimes C_B$, every $g \in V_{\leq}$ has at least one adjacent edge $e$ that is in $E_h$ and thus

$$\mathbb{E}_{g\sim G^0}\left[\mathbf{1}[g \in T] \cdot \Pr_{(g,g')\sim\pi_g(0)}[(g,g') \in E_h]\right] = \mathbb{E}_{(g,g')\sim\pi(1)}[\mathbf{1}[g \in T] \cdot \mathbf{1}[(g,g') \in E_h]] \geq \frac{1}{2\Delta} \cdot wt_0(V_{\leq}).$$

On the other hand,

$$\mathbb{E}_{g\sim G^0}\left[\mathbf{1}[g \in T] \cdot \Pr_{(g,g')\sim\pi_g(0)}[(g,g') \in E_h]\right] = wt_0(T) \cdot \mathbb{E}_{g\in T}\left[\Pr_{(g,g')\sim\pi_g(0)}[(g,g') \in E_h]\right].$$

Shorthand $\mathbb{E}_{g\in T}[\Pr_{(g,g')\sim\pi_g(0)}[(g,g') \in E_h]]$ as $\mu_h$ and combine this equation with the first inequality to get

$$\mu_h \geq \frac{wt_0(V_{\leq})}{2\Delta \cdot wt_0(T)} \geq \frac{\delta^2}{42}.$$

Therefore

$$\mu_h \leq \frac{\delta^2}{84} \cdot \Pr_{g\in T}\left[\Pr_{(g,g')\sim\pi_g(0)}[(g,g') \in E_h] \leq \frac{\delta^2}{84}\right] + \Pr_{g\in T}\left[\Pr_{(g,g')\sim\pi_g(0)}[(g,g') \in E_h] > \frac{\delta^2}{84}\right]$$
$$\frac{\delta^2}{168} \leq \Pr_{g\in T}\left[\Pr_{(g,g')\sim\pi_g(0)}[(g,g') \in E_h] > \frac{\delta^2}{84}\right].$$

□

We also agrue that $g \in T$ on average does not have too many neighbors in $V_>$.

**Claim 23.4.** *Let $X$ and $f$ satisfy the conditions in Claim 23.1. Then at most $\frac{\delta^2}{200}$-fraction of vertices in $T$ has at least $\frac{800}{\delta^2} \cdot \frac{1}{\Delta^{\frac{1}{2}-\varepsilon}}$-fraction of the neighbors in $V_>$.*

*Proof.* Again denote by $\mu_> = \mathbb{E}_{g \in T}[\Pr_{(g,g') \sim \pi_g(0)}[g' \in V_>]]$. Then by spectral decomposition we have that

$$wt_0(T) \cdot \mu_> = \left\langle \vec{1}_T, M_0^+ \vec{1}_{V_>} \right\rangle \leq wt_0(T) \cdot wt_0(V_>) + \frac{2}{\sqrt{\Delta - 1}} \cdot \sqrt{wt_0(T) \cdot wt_0(V_>)}.$$

Plugging in the bound of $wt(V_>)$ from Claim 23.1 we get that $\mu_> \leq \frac{4}{\Delta^{\frac{1}{2}-\varepsilon}}$. Then applying the Markov's inequality yields the claim statement. □

Now we get that with probability $\frac{\delta^2}{168} - \frac{\delta^2}{200}$, a $g \in T$ has

1. at least $\frac{\delta^2}{84}$-fraction of the adjacent edges in $E_h$ and,
2. at most $\frac{800}{\delta^2} \cdot \frac{1}{\Delta^{\frac{1}{2}-\varepsilon}}$-fraction of the neighbors in $V_>$.

Pick any such $g \in T$. Every adjacent edge $(g, ag) \in E_h$ satisfies that

$$\left\| f|_{X_{g,ag}(0)} \right\| \geq \delta - \frac{1}{\delta \Delta^{\frac{1}{2}+\varepsilon}}, \text{ and } \operatorname{dist}(f|_{X_{g,ag}(0)}, C_B) \leq \frac{1}{\delta \Delta^{\frac{1}{2}-\varepsilon}}.$$

An analogous statement holds for every adjacent edge $(g, gb) \in E_h$.

For every adjacent edge $(g, g') \notin E_h$ but $g' \in V_\leq$ it holds that

$$\left\| f|_{X_{(g,g')}(0)} \right\| \leq \frac{1}{\delta \Delta^{\frac{1}{2}+\varepsilon}}.$$

Therefore we have

$$\begin{aligned} \operatorname{dist}(f|_{X_g(1)}, C_A \otimes \mathbb{F}^\Delta) + \operatorname{dist}(f, \mathbb{F}^\Delta \otimes C_B) &\leq 2 \cdot \left( \Pr_{(g,g') \sim \pi_g(0)}[g' \in V_>] \cdot 1 + \Pr_{(g,g') \sim \pi_g(0)}[g' \in V_\leq] \cdot \frac{1}{\delta \Delta^{\frac{1}{2}+\varepsilon}} \right) \\ &= \Theta\left( \frac{1}{\delta^2 \Delta^{\frac{1}{2}-\varepsilon}} \right), \end{aligned}$$

and

$$\begin{aligned} \left\| f|_{X_g(1)} \right\| &\geq \Pr_{(g,g') \sim \pi_g(0)}[(g,g') \in E_h] \cdot \left( \delta - \frac{1}{\delta \Delta^{\frac{1}{2}+\varepsilon}} \right) \\ &\geq \Theta\left(\delta^3\right). \end{aligned}$$

Since $C_A \otimes C_B$ is $\frac{1}{\Delta^{\frac{1}{2}+\varepsilon}}$-robust locally testable, we have that

$$\operatorname{dist}(f|_{X_g(1)}, C_A \otimes C_B) \leq \Theta(1) \cdot \left( \operatorname{dist}(f|_{X_g(1)}, C_A \otimes \mathbb{F}^\Delta) + \operatorname{dist}(f|_{X_g(1)}, \mathbb{F}^\Delta \otimes C_B) \right) = \Theta\left( \frac{1}{\delta^2 \Delta^{\frac{1}{2}-\varepsilon}} \right).$$

Therefore there exists some nonzero codeword $c_g \in C_A \otimes C_B$ such that that

$$\operatorname{dist}(f|_{X_g(1)}, c_g) = \Theta\left( \frac{1}{\delta^2 \Delta^{\frac{1}{2}-\varepsilon}} \right) \ll \left\| f|_{X_g(1)} \right\|.$$

So if we extend $c_g$ to a global function $\bar{c}_g$ by assigning zeros to all squares not in $X_g(1)$, then

$$\| f - \bar{c}_g \| < \| f \|.$$

Thus we complete the proof.