

# 1 Introduction

Expander graphs are families of sparse graphs that nevertheless exhibit strong connectivity properties. In the classical, regular setting, one works with a fixed integer  $d \geq 3$  and considers a family of finite  $d$ -regular graphs  $\{G_n\}$  with  $|V(G_n)| \rightarrow \infty$ . The Alon-Boppana bound shows that there is a universal obstruction to making  $\lambda_2(G_n)$ , the second largest eigenvalue of the adjacency matrix of  $G_n$ , too small:

$$\lambda_2(G_n) \geq 2\sqrt{d-1} - o_n(1). \quad (1)$$

Thus, for fixed  $d$  and any  $\epsilon > 0$ , no infinite family of  $d$ -regular graphs can satisfy  $\lambda_2(G_n) \leq 2\sqrt{d-1} - \epsilon$ . Graphs whose nontrivial eigenvalues all lie in the interval  $[-2\sqrt{d-1}, 2\sqrt{d-1}]$  are called *Ramanujan graphs* and are regarded as optimally expanding.

The constant  $2\sqrt{d-1}$  is not random; it is the spectral radius of the infinite  $d$ -regular tree  $T_d$ , the *universal covering* (defined later) graph of every finite  $d$ -regular graph. Let  $A_{T_d}$  be the adjacency operator on the Hilbert space  $\ell^2(V(T_d))$ . Its *spectral radius*, akin to that of a finite matrix, is defined by the Rayleigh quotient

$$\rho(T_d) = \sup_{0 \neq f \in \ell^2(V(T_d))} \frac{\langle f, A_{T_d} f \rangle}{\langle f, f \rangle}. \quad (2)$$

Kesten [4] showed that  $\rho(T_d) = 2\sqrt{d-1}$ . In this notation, the Alon-Boppana bound can be re-written as

$$\lambda_2(G_n) \geq \rho(T_d) - o_n(1), \quad (3)$$

so the nontrivial spectrum of any large  $d$ -regular graph must accumulate at the spectral radius of its universal cover. The goal of this write-up is to understand the corresponding picture for *irregular* graphs. When the degree is not constant, there is no single parameter  $d$  and the expression  $2\sqrt{d-1}$  has no direct meaning. However, it can be proven that every finite connected graph  $G$  still has a universal cover  $T$ , and consequently the notion  $\rho(T)$  is still well-defined. The **Greenberg-Lubotzky** theorem [2] is a generalization of the Alon-Boppana bound to irregular graphs. It asserts that for any family of finite graphs whose universal covers all coincide with the same tree  $T$ , the nontrivial eigenvalues of their adjacency matrices cannot avoid  $\rho(T)$ . In other words,  $\rho(T)$  plays exactly the role  $\rho(T_d)$  plays in the regular case; it is the universal lower bound for how small the nontrivial spectrum can be in a large family of quotients of  $T$ .

## 2 Preliminaries: Universal Covers & Spectral Radius

We first collect some basic objects that will appear in the Greenberg-Lubotzky theorem: graph coverings, the universal cover of a finite graph, and the spectral radius of the adjacency operator on this infinite cover. Throughout, we adopt notations used throughout the CS590 lecture notes [6]: all graphs in this note are finite, connected, undirected, and without loops or multiple edges, unless stated otherwise. For  $G = (V, E)$  we write  $|V| = n$ . We usually assume that the degrees are uniformly bounded, so that all constructions below are well behaved.

### 2.1 Graphs & Adjacency Operators

The (combinatorial) adjacency operator of  $G$  is the linear map

$$A_G : \mathbb{R}^V \rightarrow \mathbb{R}^V : \quad (A_G f)(v) = \sum_{(u,v) \in E} f(u), \quad (4)$$

where the sum runs over neighbors  $u$  of  $v$ . When  $V = \{1, \dots, n\}$ , this is the usual  $n \times n$  adjacency matrix. Since  $A_G$  is real symmetric, it admits an orthonormal basis of eigenvectors and has real eigenvalues  $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ . For a connected  $d$ -regular graph one has  $\lambda_1(G) = d$ ; in that setting, the *nontrivial spectrum* refers to  $\lambda_2(G), \dots, \lambda_n(G)$ , and the Alon-Boppana bound shows that, for any infinite family of such graphs, the nontrivial eigenvalues must accumulate at  $2\sqrt{d-1}$ .

In the irregular case, the top eigenvalue and its eigenvector no longer have this simple description, and it is more natural to compare the spectrum of  $G$  with the spectrum of an infinite graph associated to it: its *universal cover*.

## 2.2 Coverings & the Universal Cover

**Definition 2.1** (Covering Map). Let  $G = (V, E)$  and  $\tilde{G} = (\tilde{V}, \tilde{E})$  be connected graphs. A map  $\pi : \tilde{V} \rightarrow V$  is called a **covering map** if (i) it is surjective, and (ii) for every vertex  $\tilde{v} \in \tilde{V}$ , the restriction of  $\pi$  induces a bijection between the neighbors of  $\tilde{v}$  in  $\tilde{G}$  and the neighbors of  $\pi(\tilde{v})$  in  $G$ .

In this situation,  $\tilde{G}$  is called a **cover** of  $G$ , and  $G$  is called a **quotient** of  $\tilde{G}$ .

The simplest example is the cyclic cover: the cycle  $C_{2m}$  covers  $C_m$  via the map that sends vertices  $i$  and  $i + m$  to of the larger  $2m$ -cycle to vertex  $i$  of the smaller  $m$ -cycle. The collection of all covers of a fixed connected graph  $G$  forms a partially ordered set under the relation “covers;” among them, there is a maximal object, the *universal cover*, which we formalize now.

**Definition 2.2** (Universal Cover). Let  $G$  be a connected finite graph. A **universal cover** of  $G$  is a tree  $T$  equipped with a covering map  $T \rightarrow G$  such that for any other  $\pi' : H \rightarrow G$ , there exists a unique covering map  $\varphi : T \rightarrow H$  such that  $\pi = \pi' \circ \varphi$ .

It is well known that the universal cover exists and is unique up to graph isomorphism [5]. In particular, any connected finite graph admits a canonical infinite tree  $T$  that captures all of its finite covering graphs simultaneously. For the purposes of the Greenberg-Lubotzky theorem, it is convenient to have a concrete model of  $T$  in terms of walks.

**Definition 2.3** (Non-backtracking Walks & the Universal Cover). Fix a base vertex  $v_0 \in V(G)$ . A walk  $w = (v_0, v_1, \dots, v_k)$  is called **non-backtracking** if  $v_{i+2} \neq v_i$  whenever the two vertices are defined. That is, the walk never *immediately* traverses an edge then returns across it.

For a fixed  $v_0 \in V(G)$ , the universal cover  $T$  of  $G$  can be realized as follows.

- (Vertices) The vertex set  $V(T)$  consists of all non-backtracking walks in  $G$  starting at  $v_0$ .
- (Edges) Two walks are adjacent in  $T$  if one walk is a single-step extension of the other.
- (Covering map) The covering map  $\pi : T \rightarrow G$  sends a walk to its terminal vertex.

In this model, the tree structure of  $T$  is transparent: extending a walk by one step cannot create a cycle in  $T$ , and every vertex in  $T$  has exactly one “parent” and finitely many “children” determined by its endpoint in the original graph  $G$ . If  $G$  is  $d$ -regular, every vertex in  $T$  has degree  $d$ ; the universal cover in this case is then the infinite  $d$ -regular tree, denoted  $T_d$ .

Finally, for convenience we fix notation for local neighborhoods by drawing connections to real analysis. If  $X$  is a graph and  $x \in V(X)$ , the **ball** of radius  $r$  around  $x$ , denoted  $B_X(x, r)$  or  $B(x, r)$ , is the induced subgraph spanned by all vertices at graph distance at most  $r$  from  $x$ . The main feature of a covering map  $\pi : T \rightarrow G$  is that *every ball*  $B_T(\tilde{v}, r)$  *maps isomorphically onto the ball*  $B_G(\pi(\tilde{v}), r)$ . In particular, all finite-radius neighborhoods in  $T$  look like “unwrapped” versions of neighborhoods in  $G$ .

## 2.3 Spectral Radius of the Universal Cover

The universal cover  $T$  is generally infinite, so its adjacency operator acts on an infinite-dimensional space. Let  $V(T)$  be the vertex set of the universal cover. Consider the Hilbert space  $\ell^2(V(T))$  of square-summable real functions on  $V(T)$ ,

$$\ell^2(V(T)) = \left\{ f : V(T) \rightarrow \mathbb{R} : \sum_{x \in V(T)} f(x)^2 < \infty \right\} \quad (5)$$

equipped with the usual inner product  $\langle f, g \rangle = \sum_x f(x)g(x)$ . The adjacency operator

$$A_T = \ell^2(V(T)) \rightarrow \ell^2(V(T)), \quad (A_T f)(x) = \sum_{y \sim x} f(y) \quad (6)$$

is a bounded self-adjoint operator whenever  $T$  has uniformly bounded degrees. Its operator norm is the natural notion of “spectral radius” for the universal cover. (The notation  $y \sim x$  means  $y$  is adjacent to  $x$  in the universal cover  $T$ .)

**Definition 2.4** (Spectral Radius of the Universal Cover). The spectral radius of  $T$  is defined as

$$\rho(T) = \|A_T\|_{\text{op}} = \sup_{0 \neq f \in \ell^2(V(T))} \frac{\langle f, A_T f \rangle}{\langle f, f \rangle}. \quad (7)$$

In the regular case, this reproduces the constant appearing in the Alon-Boppana bound: Kesten [4] showed that the spectrum of  $A_{T_d}$  is the interval  $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ ; in particular  $\rho(T_d) = 2\sqrt{d-1}$ . For an arbitrary finite graph  $G$  with universal cover  $T$ , the Greenberg-Lubotzky theorem will assert that  $\rho(T)$  plays the same role: it is a universal asymptotic lower bound. We present this result following Hoory-Linial-Wigderson in the next section.

### 3 The Greenberg-Lubotzky Theorem

Let  $G$  be a finite connected graph with adjacency matrix  $A_G$ . Write the eigenvalues of  $A_G$  as  $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ , and recall that  $\lambda_1(G)$  is simple and has a positive eigenvector by Perron-Frobenius.

**Definition 3.1** (Nontrivial Spectral Radius). The **nontrivial spectral radius** of  $G$  is  $\lambda(G) = \max_{2 \leq i \leq n} |\lambda_i(G)|$ . This is the quantity that governs expansion.

**Theorem 3.2** (Greenberg-Lubotzky). *Let  $\{G_i\}_{i=1}^\infty$  be a family of finite connected graphs with  $|V(G_i)| \rightarrow \infty$ , all covered by the same universal cover  $T$ . Then,*

$$\liminf_{i \rightarrow \infty} \lambda(G_i) \geq \rho(T). \quad (8)$$

*Equivalently, for every  $\epsilon > 0$ , all but finitely many graphs  $G_i$  satisfy  $\lambda(G_i) \geq \rho(T) - \epsilon$ . This generalizes the Alon-Boppana lower bound from the  $d$ -regular case, where  $T = T_d$  and  $\rho(T_d) = 2\sqrt{d-1}$ .*

**Corollary 3.3** (Alon-Boppana, restated). *Let  $\{G_i\}_{i=1}^\infty$  be finite, connected,  $d$ -regular graphs with  $|V(G_i)| \rightarrow \infty$ . Then,*

$$\liminf_{i \rightarrow \infty} \lambda_2(G_i) \geq 2\sqrt{d-1}. \quad (9)$$

The proof of Theorem 3.2 is rather long and divided into multiple parts. For consistency, we follow the outline sketched in [3] and [1], and indicate overlap with lecture materials when applicable.

#### 3.1 Rayleigh Quotients & Closed Walks in Finite Graphs

Let  $G$  be a finite connected graph with adjacency matrix  $A$ . For each integer  $\ell \geq 0$ , the  $(u, v)$  entry of  $A^\ell$  counts the number of length- $\ell$  walks from  $u$  to  $v$ . Consequently,  $(A^{2k})_{u,u} = C_G(2k, u)$  counts the number of closed, length- $2k$  walks starting and ending at  $u$ . We will need the variational description of  $\lambda(G)$  in terms of Rayleigh quotients and the “two-point” test vector used in the proof of Alon-Boppana [7].

**Lemma 3.4** (Rayleigh Quotient for  $\lambda(G)$ ). *Let  $G, A$  be given with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Let  $v_1$  be the unit Perron eigenvector corresponding to  $\lambda_1(G)$ . Then, for every integer  $k \geq 1$ ,*

$$\lambda(G)^{2k} = \max \left\{ \frac{f^\top A^{2k} f}{\|f\|^2} : f \neq 0, f \perp v_1 \right\}. \quad (10)$$

*Proof.* Since  $A$  is real symmetric, we can choose an orthonormal eigenbasis  $\{v_1, \dots, v_n\}$  with  $Av_j = \lambda_j v_j$ . Write  $f$  as  $f = \sum_{j=1}^n a_j v_j$ . Then  $f \perp v_1$  if and only if  $a_1 = 0$ . Moreover,  $A^{2k} f = \sum_{j=1}^n a_j \lambda_j^{2k} v_j$ , so

$$\frac{f^\top A^{2k} f}{\|f\|^2} = \frac{\sum_{j=1}^n |a_j|^2 \lambda_j^{2k}}{\sum_{j=1}^n |a_j|^2}, \quad (11)$$

If  $a_1 = 0$ , this becomes  $(\sum_{j \geq 2} |a_j|^2 \lambda_j^{2k}) / (\sum_{j \geq 2} |a_j|^2)$ , whose maximum over the choices of  $(a_j)_{j \geq 2}$  is clearly  $\max_{j \geq 2} |\lambda_j|^{2k} = \lambda(G)^{2k}$ .  $\square$

**Lemma 3.5** (Two-point Test Vector [7]). *Let  $G$  be a finite graph of diameter  $\Delta$ , and let  $s, t \in V(G)$  satisfy  $d_G(s, t) = \Delta$ . Let  $A$  be the adjacency matrix and  $v_1$  the (unit) Perron eigenvector. Fix an integer  $k$  with  $2k < \Delta$ , and define  $f = v_1(t)e_s - v_1(s)e_t$ , where  $e_x$  is the standard basis vector for  $x$ . Then  $f \perp v_1$ , and*

$$\lambda(G)^{2k} \geq \frac{f^\top A^{2k} f}{\|f\|^2} \geq \min(C_G(2k, s), C_G(2k, t)). \quad (12)$$

*Proof.* Write  $a = v_1(t)$  and  $b = v_1(s)$ . Then  $\|f\|^2 = a^2 + b^2$ ,  $\langle f, v_1 \rangle = ab - ba = 0$ , so  $f \perp v_1$ .

Since  $(A^{2k})_{s,t}$  counts the number of length- $2k$  walks from  $s$  to  $t$ , given that  $2k < d_G(s, t) = \Delta$ , there is no such walk, so  $(A^{2k})_{s,t} = 0$ . Thus expanding  $f^\top A^{2k} f$  gives

$$\begin{aligned} f^\top A^{2k} f &= (ae_s - be_t)^\top A^{2k} (ae_s - be_t) = a^2(A^{2k})_{s,s} + b^2(A^{2k})_{t,t} - 2ab(A^{2k})_{s,t} \\ &= a^2 C_G(2k, s) + b^2 C_G(2k, t) \end{aligned} \quad (13)$$

and consequently

$$\frac{f^\top A^{2k} f}{\|f\|^2} = \frac{a^2}{a^2 + b^2} C_G(2k, s) + \frac{b^2}{a^2 + b^2} C_G(2k, t) \geq \min(C_G(2k, s), C_G(2k, t)). \quad (14)$$

Applying Lemma 3.4 concludes the proof.  $\square$

## 3.2 Coverings and Closed Walk Counts

Now let  $\pi : T \rightarrow G$  be a covering map from an infinite tree  $T$  (e.g. the universal cover) onto a finite connected graph  $G$ . We will show that closed-walk counts do not decrease when passing to a quotient in the following lemma.

**Lemma 3.6** (Closed Walks Inject Under the Projection). *Let  $\pi : T \rightarrow G$  be a covering map,  $u \in V(G)$ , and  $\tilde{u} \in V(T)$  with  $\pi(\tilde{u}) = u$ . Then, for each length  $\ell \geq 0$ :*

- Every walk  $w = (u = v_0, \dots, v_\ell)$  in  $G$  has a unique lift  $\tilde{w} = (\tilde{u} = \tilde{v}_0, \dots, \tilde{v}_\ell)$  in  $T$  with  $\pi(\tilde{v}_i) = v_i$  for all  $i$ ;
- The map  $\Phi_\ell$  mapping closed walks of length  $\ell$  at  $\tilde{u}$  to closed walks of length  $\ell$  at  $u$ , given by projecting each vertex via  $\pi$ , is injective.

*In particular, for every even length  $2k$ , we have  $C_G(2k, u) \geq C_T(2k, \tilde{u})$ .*

*Proof.* For (i), we prove by induction on  $\ell$  that every walk in  $G$  starting at  $u$  lifts uniquely once we fix the starting lift  $\tilde{u}$ . For  $\ell = 0$  there is nothing to prove. Suppose the claim holds for length  $\ell$  and let  $w = (u = v_0, \dots, v_\ell, v_{\ell+1})$  be a walk of length  $\ell + 1$ . By induction its prefix  $(v_0, \dots, v_\ell)$  has a unique lift  $(\tilde{v}_0 = \tilde{u}, \dots, \tilde{v}_\ell)$ . Since  $v_{\ell+1}$  is a neighbor of  $v_\ell$  and  $\pi$  is a bijection, there is a unique neighbor  $\tilde{v}_{\ell+1}$  of  $\tilde{v}_\ell$  with  $\pi(\tilde{v}_{\ell+1}) = v_{\ell+1}$ . This extends the lift uniquely.

For (ii), let  $\tilde{w}_1, \tilde{w}_2$  be two closed walks at  $\tilde{u}$  of length  $\ell$  with the same projection  $w$  in  $G$ . Then both  $\tilde{w}_1, \tilde{w}_2$  are lifts of the same walk starting at  $\tilde{u}$ . By uniqueness of lifts,  $\tilde{w}_1 = \tilde{w}_2$ . Thus  $\Phi_\ell$  is injective. This immediately implies  $C_G(2k, u) \geq C_T(2k, \tilde{u})$ .  $\square$

## 3.3 Closed Walks and the Spectral Radius on Universal Cover

Let  $T$  be the universal cover, with adjacency operator  $A_T : \ell^2(V(T)) \rightarrow \ell^2(V(T))$ . For each vertex  $v \in V(T)$  and integer  $k \geq 0$ , denote by  $C_T(2k, v)$  the number of closed length- $2k$  walks starting at  $v$  as usual, so that  $C_T(2k, v) = \langle \delta_v, A_T^{2k} \delta_v \rangle$ . Define  $M_{2k} = \max_{v \in V(T)} C_T(2k, v)$ . We need to show that the exponential growth rate of these quantities recovers the spectral radius  $\rho(T)$ .

**Lemma 3.7** (Spectral Radius from Closed Walks). *With notations as above, for every vertex  $v \in V(T)$  we have*

$$\lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} = \rho(T), \quad (15)$$

and in particular  $\lim_{k \rightarrow \infty} (M_{2k})^{1/2k} = \rho(T)$ .

*Proof.* We first prove the upper bound. For every  $v \in V(T)$ ,

$$C_T(2k, v) = \langle \delta_v, A_T^{2k} \delta_v \rangle \leq \|A_T^{2k}\|_{\text{op}} \|\delta_v\|^2 = \rho(T)^{2k}, \quad (16)$$

so  $\limsup_{k \rightarrow \infty} C_T(2k, v)^{1/2k} \leq \rho(T)$  for every  $v$ , and hence  $\limsup_{k \rightarrow \infty} (M_{2k})^{1/2k} \leq \rho(T)$  as well.

For the lower bound, we use the spectral theorem for bounded self-adjoint operators. There exists a projection-valued measure  $E$  on  $\mathbb{R}$  with support equal to the spectrum of  $A_T$ , and  $A_T = \int \lambda \, dE(\lambda)$ . For each vertex  $v$ , define the scalar spectral measure

$$\mu_v(B) = \langle E(B)\delta_v, \delta_v \rangle, \quad \text{for } B \subset \mathbb{R} \text{ Borel.} \quad (17)$$

Then  $\mu_v$  is a probability measure and

$$C_T(2k, v) = \langle \delta_v, A_T^{2k} \delta_v \rangle = \int \lambda^{2k} \, d\mu_v(\lambda). \quad (18)$$

Let  $\beta_v = \sup\{|\lambda| : \lambda \in \text{supp}(\mu_v)\}$ . We first claim that for fixed  $v$ ,  $\lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} = \beta_v$ . The proof is twofold:

- Since  $|\lambda| \leq \beta_v$  on the support of  $\mu_v$ ,

$$C_T(2k, v) \leq \int |\lambda|^{2k} \, d\mu_v(\lambda) \leq \beta_v^{2k}, \quad (19)$$

so  $\limsup_{k \rightarrow \infty} C_T(2k, v)^{1/2k} \leq \beta_v$ .

- Conversely, fix  $\epsilon > 0$ . By definition of  $\beta_v$ , the set  $S_\epsilon = \{|\lambda| \geq \beta_v - \epsilon\}$  has positive  $\mu_v$ -measure, say  $\mu_v(S_\epsilon) = \alpha > 0$ . For all  $k$ ,

$$C_T(2k, v) = \int \lambda^{2k} \, d\mu_v(\lambda) \geq \int_{S_\epsilon} |\lambda|^{2k} \, d\mu_v(\lambda) \geq \alpha(\beta_v - \epsilon)^{2k}. \quad (20)$$

Hence  $\liminf_{k \rightarrow \infty} C_T(2k, v)^{1/2k} \geq \beta_v - \epsilon$ , and letting  $\epsilon \rightarrow 0$  shows  $\liminf_{k \rightarrow \infty} C_T(2k, v)^{1/2k} \geq \beta_v$ .

Combining the two inequalities yields  $\lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} = \beta_v$  for each  $v$ .

Next we show that the numbers  $\beta_v$  do not depend on  $v$ . Let  $u, v \in V(T)$ , and let  $\ell$  be the length of some fixed simple path from  $u$  to  $v$  in  $T$ . Any closed walk of length  $2k$  at  $v$  gives rise to a closed walk of length  $2k + 2\ell$  at  $u$  by walking from  $u$  to  $v$  along this path, traversing the closed walk at  $v$ , and then returning along the path. This construction is injective, so

$$C_T(2k + 2\ell, u) \geq C_T(2k, v) \quad \text{for all } k \geq 0. \quad (21)$$

Taking  $(2k)$ -th roots, we obtain

$$C_T(2k, v)^{1/2k} \leq C_T(2k + 2\ell, u)^{1/2k} = \left( C_T(2k + 2\ell, u)^{1/(2k+2\ell)} \right)^{(2k+2\ell)/(2k)}. \quad (22)$$

Taking  $\limsup_{k \rightarrow \infty}$  on both sides and using that  $(2k + 2\ell)/(2k) \rightarrow 1$  gives

$$\beta_v = \lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} \leq \limsup_{k \rightarrow \infty} C_T(2k + 2\ell, u)^{1/(2k+2\ell)} = \beta_u. \quad (23)$$

Reversing the roles of  $u$  and  $v$  shows  $\beta_u \leq \beta_v$ , hence  $\beta_u = \beta_v$  for all  $u, v \in V(T)$ . In particular, there is a constant  $\beta$  such that  $\beta_v = \beta$  for all  $v$ , and

$$\lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} = \beta \quad \text{for all } v \in V(T). \quad (24)$$

On the other hand, by definition,  $\rho(T) = \sup\{|\lambda| : \lambda \in \text{spec}(A_T)\}$ , so the spectrum of  $A_T$  is the closure of the union of the supports of the measures  $\mu_v$ , so  $\rho(T) = \sup_v \beta_v = \beta$ . It follows that for every  $v$ ,

$$\lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} = \rho(T), \quad (25)$$

which proves the first assertion (Equation (15)). Finally, since  $M_{2k} = \max_v C_T(2k, v)$ , we have  $M_{2k} \geq C_T(2k, v)$  for each fixed  $v$ . Picking any  $v$  and using the limit above,

$$\liminf_{k \rightarrow \infty} (M_{2k})^{1/2k} \geq \lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} = \rho(T). \quad (26)$$

Combined with the bound on  $M_{2k}$  from the beginning of the proof, we see  $\lim_{k \rightarrow \infty} (M_{2k})^{1/2k} = \rho(T)$  as claimed.  $\square$

### 3.4 Proof of the Greenberg-Lubotzky Theorem

We now assemble the ingredients to prove Theorem 3.2.

*Proof of Theorem 3.2.* Let  $\{G_i\}$  be as in the statement: each  $G_i$  is finite, connected, and there is a covering map  $\pi_i : T \rightarrow G_i$ . Let  $A_i$  be the adjacency matrix of  $G_i$  and  $\lambda(G_i)$  its largest nontrivial eigenvalue in absolute value.

Because  $T$  has bounded degrees (since it covers a finite graph), the number of vertices in a ball of radius  $r$  in  $T$  grows at most exponentially in  $r$ . For each  $i$ , let  $\Delta_i$  be the diameter of  $G_i$ . Since  $G_i$  is the quotient of  $T$  with  $|V(G_i)| \rightarrow \infty$ , we necessarily have  $\Delta_i \rightarrow \infty$ . Otherwise, all  $G_i$  would sit inside a fixed ball of  $T$  and could not be infinitely many distinct finite quotients.

Fix  $i$  and abbreviate  $G_i, A_i, \Delta_i$  as  $G, A, \Delta$ , respectively. Choose vertices  $s, t \in V(G)$  with  $d_G(s, t) = \Delta$ . Set  $k = k_i = \lfloor (\Delta - 1)/2 \rfloor$  so that  $2k < \Delta$ . By Lemma 3.5 we have

$$\lambda(G)^{2k} \geq \min(C_G(2k, s), C_G(2k, t)). \quad (27)$$

We next compare closed paths in  $G$  and in  $T$ . For each of  $s, t$ , choose a lift  $\tilde{s}, \tilde{t} \in V(T)$  with  $\pi_i(\tilde{s}) = s$  and  $\pi_i(\tilde{t}) = t$ . By Lemma 3.6,  $C_G(2k, s) \geq C_T(2k, \tilde{s})$  and  $C_G(2k, t) \geq C_T(2k, \tilde{t})$ , so

$$\lambda(G)^{2k} \geq \min(C_T(2k, \tilde{s}), C_T(2k, \tilde{t})). \quad (28)$$

Now fix  $\epsilon > 0$ . By Lemma 3.7, for every vertex  $v \in V(T)$  we have  $\lim_{k \rightarrow \infty} C_T(2k, v)^{1/2k} = \rho(T)$ . Since  $T$  is the universal cover of a finite graph, there are only finitely many orbits of vertices under its deck transformation group (one for each vertex of the base graph). Closed-walk counts  $C_T(2k, v)$  are constant on each orbit, so the convergence in Lemma 3.7 is uniform over  $v \in V(T)$ : for our fixed  $\epsilon > 0$  there exists  $K$  such that

$$C_T(2k, v) \geq (\rho(T) - \epsilon)^{2k} \quad \text{for all } v \in V(T), k \geq K. \quad (29)$$

(Indeed, for each orbit choose a representative vertex and a corresponding  $K$ , and take the maximum over finitely many orbits.) For sufficiently large  $i$  we have  $k_i \geq K$ , and then

$$C_T(2k_i, \tilde{s}), C_T(2k_i, \tilde{t}) \geq (\rho(T) - \epsilon)^{2k_i}, \quad (30)$$

so

$$\lambda(G_i)^{2k_i} \geq \min(C_T(2k_i, \tilde{s}), C_T(2k_i, \tilde{t})) \geq (\rho(T) - \epsilon)^{2k_i}. \quad (31)$$

Taking  $(2k_i)^{\text{th}}$  roots yields  $\lambda(G_i) \geq \rho(T) - \epsilon$  for all sufficiently large  $i$ . Since  $\epsilon > 0$  was arbitrary, this shows

$$\liminf_{i \rightarrow \infty} \lambda(G_i) \geq \rho(T), \tag{32}$$

completing the proof of the Greenberg-Lubotzky theorem. □

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