Due date: October 30, 2025

Problem 1: Let S be a set of n points in $\mathbb{R}^2$, such that each point $p \in S$ has a positive weight $w(p)$. We define the weighted distance of an arbitrary point $x \in \mathbb{R}^2$ from p to be $d(x, p) := \|x - p\| w(p)$. The weighted Voronoi cell of a point $p \in S$ is as usual defined as $\text{Vor}(p) := \{x \in \mathbb{R}^2 \mid d(x, p) \leq d(x, p') \, \forall p' \in S\}$. The resulting diagram is known as the *multiplicative weighted Voronoi diagram*.

(i) Describe the bisector of two points in the multiplicative weighted Voronoi diagram.

(ii) What is the complexity of the multiplicative weighted Voronoi diagram? Justify your answer.

Problem 2: A k-clustering of a set P of n points in the plane is a partitioning of P into k non-empty subsets $P_1, \ldots, P_k$. Define the distance $d(P_i, P_j)$ between any pair P_i, P_j of clusters to be the minimum distance between one point from P_i and one point from P_j, that is,

$$d(P_i, P_j) := \min_{p \in P_i,\, q \in P_j} \|p - q\|.$$

We want to find a k-clustering (for given k and P) that maximizes the minimum distance between clusters.

(i) Suppose the mimimum distance between clusters is achieved by points $p \in P_i$ and $q \in P_j$. Prove that pq is an edge of the Delaunay triangulation of P.

(ii) Give an $O(n \log n)$ time algorithm to compute a k-clustering maximizing the minimum distance between clusters. (**Hint:** *Use a Union-Find data structure.*)

Problem 3: Let L be a set of n lines in $\mathbb{R}^2$. The *level* of a point $x \in \mathbb{R}^2$, denoted by $\lambda(x)$, is the number of lines of L that lie below x. Suppose we choose a random subset $R \subseteq L$ by choosing each line of L with probability $\frac{1}{k}$ for some $k \in [1, n]$. Let $V_j(L)$ denote the set of vertices of $A(L)$ of level j.

(i) What is the probability of a level-j vertex of $A(L)$ being a vertex of $V_0(R)$?

(ii) Use (i) to prove that $|V_{\leq k}(L)| = \sum_{j=0}^{k} |V_j(L)| = O(nk)$. (**Hint:** *obtain a lower bound on* $\mathbb{E}\left[|V_0(R)|\right]$ *in terms of* $|V_{\leq k}(L)|$ *using (i)*)

Problem 4: Let L be a set of n lines and P a set of m points in $\mathbb{R}^2$. Describe an $O(\min\{mn^{1/2}, nm^{1/2}\} \log n)$ time algorithm to detect whether any point of P lies on any line of L.

Problem 5: Let P be a set of n points in $\mathbb{R}^2$. Describe an $O(n^2)$ time algorithm to compute the minimum area triangle spanned by P, i.e., return $\arg\min_{p \neq q \neq r \in P} \text{Area}(\Delta pqr)$.