

Lemma 0.1. Fix real-valued functions X, Y on the electorate V . For any real-valued function Z on V , in order for (X, Y, Z) to be realized by some metric d via ??, it is necessary and sufficient that

$$Z(v) \geq Z_{\min}(v) = \max \left\{ \begin{array}{l} 0, X(v), -Y(v), \\ \frac{\|X\|_{\infty} + X(v)}{2}, \frac{\|Y\|_{\infty} - Y(v)}{2}, \frac{\|X + Y\|_{\infty} - (Y(v) - X(v))}{2} \end{array} \right\} \quad \text{for all } v. \quad (1)$$

Proof. We first prove necessity. Because d is nonnegative, we must have $d(v, C) = Z(v) \geq 0$, $d(v, A) = Z(v) - X(v) \geq 0$, and $d(v, B) = Z(v) + Y(v) \geq 0$ from ??. Triangle inequalities for (v, A, C) imply

$$|d(v, A) - d(v, C)| = |X(v)| \leq d(A, C) \leq d(v, A) + d(v, C) = 2Z(v) - X(v).$$

Taking supremum over the first \leq gives $d(A, C) \geq \|X\|_{\infty}$; combining with the second \leq ,

$$2Z(v) - X(v) \geq \|X\|_{\infty} \quad \text{so} \quad Z(v) \geq \frac{\|X\|_{\infty} + X(v)}{2}.$$

The remaining two terms can be obtained by enforcing triangle inequalities on (v, B, C) and (v, A, B) , respectively.

For sufficiency, assume Equation (1) and define $d(A, C) = \|X\|_{\infty}$, $d(B, C) = \|Y\|_{\infty}$, and $d(A, B) = \|X + Y\|_{\infty}$. Then (A, B, C) satisfy triangle inequalities, and for each voter, the inequalities established in the necessity part flip to become upper bounds: for (v, A, C) , we have

$$|X(v)| \leq d(A, C) \leq 2Z(v) - X(v)$$

and likewise for (v, B, C) and (v, A, B) , so triangle inequalities also hold among these pairs. We remarked in ?? that this suffices for proving realizability. \square

For any instance that we work with, we may assume $\mathbb{E}X + \mathbb{E}Y < 0$; otherwise, $SC(A) \leq SC(B)$ and the instance is irrelevant for finding supremum distortion. Since the function $z \mapsto [z - \mathbb{E}X]/[z + \mathbb{E}Y]$ is decreasing in z , given fixed (X, Y) , to find Z that is metric feasible while maximizing distortion, it suffices to assume that Z is defined pointwise via Equation (1). Henceforth, we will assume that any triple (X, Y, Z) we work with satisfies this form.

Lemma 0.2 (Convexification of objective). We associate every feasible instance I with a triple

$$t(I) = (x(I), y(I), z(I)) = (\mathbb{E}X, \mathbb{E}Y, \mathbb{E}Z) \in \mathbb{R}^3$$

according to Equation (1), and write distortion as $R(x, y, z) = (z - x)/(z + y)$ where $z + y > 0$. Then, given a collection $S = \{t(I)\} = \{(x(I), y(I), z(I))\}$ of instance-induced triples,

$$\sup_S R(x, y, z) = \sup_{\text{conv}(S)} R(x, y, z). \quad (2)$$

In particular, taking $\mathcal{T} = \{t(I) : I \text{ satisfies } f(AC) = 1 - \lambda, f(CB) = \lambda\}$, to find the supremum distortion over \mathcal{T} , an important step would be to identify the vertex triples of $\text{conv}(\mathcal{T})$.

Proof. Consider any line segment in the domain, parameterized as $(x(t), y(t), z(t)) = (x_0, y_0, z_0) + \lambda(a, b, c)$ for $0 \leq \lambda \leq 1$. Along this segment,

$$R(x(t), y(t), z(t)) = \frac{(z_0 - x_0) + (c - a)t}{(z_0 + y_0) + (c + b)t}$$

whose derivative over t equals

$$R'(x(t), y(t), z(t)) = \frac{(c - a)(z_0 + y_0) - (c + b)(z_0 - x_0)}{(z_0 + y_0 + (c + b)t)^2}$$

whose sign is independent of t . Therefore, R is monotone along any line segment, and it follows that optimizing R over S may be convexified into optimizing over $\text{conv}(S)$, and that the optimum is attained at an endpoint. \square

Theorem 0.3. Fix $\lambda \in (1/2, 1)$. Define the following classes of instances:

- (i) (*C-loses-all*) Let $I_1(\lambda)$ denote the collection of instances where $f(AC) = 1 - \lambda$, $f(CB) = \lambda$, and further, $AC = BC$, $CA = CB$, with $|AC| = AC_{\min}$, $|CB| = CB_{\max}$. In any such instance, C loses all deliberations in both matchings.
- (ii) (*C-wins-all*) Let $I_2(\lambda)$ denote the collection of instances where $f(AC) = 1 - \lambda$, $f(CB) = \lambda$, and further, $AC = BC$, $CA = CB$, with $|AC| = AC_{\max}$, $|CB| = CB_{\min}$. In any such instance, C wins all deliberations in both matchings.

Then, to find $\sup SC(A)/SC(B)$ subject to $f(AC) = 1 - \lambda$, $f(CB) = \lambda$, it suffices to inspect $I_1(\lambda)$ and $I_2(\lambda)$, i.e.,

$$\sup_{\substack{f(AC)=1-\lambda \\ f(CB)=\lambda}} \frac{SC(A)}{SC(B)} = \max \left\{ \sup_{I_1(\lambda)} \frac{SC(A)}{SC(B)}, \sup_{I_2(\lambda)} \frac{SC(A)}{SC(B)} \right\}. \quad (3)$$

The proof of this theorem is rather lengthy and thus divided into several lemmas. First, by [Lemma 0.2](#), it suffices to identify the instances I whose corresponding triple $t(I)$ lies at the vertices of $\text{conv}(\mathcal{T})$. The next two lemmas help eliminate every instance failing both (i) and (ii) as potential candidates of being extremal.

Throughout this section, we will reuse two technical assumptions that are proven harmless:

- Given an instance, the distributions of X, Y, Z according to ?? obeys [Equation \(1\)](#) pointwise.
- The product extension assumption from [§5.3](#).

Lemma 0.4 (No voter ranks C second). *Suppose in instance I , the set of voters who rank C second is not of measure zero. Then $t(I)$ is the midpoint of two distinct feasible triples in \mathcal{T} ; consequently, $t(I)$ is not a vertex of $\text{conv}(\mathcal{T})$.*

Proof. A voter who ranks C second either belong to $AC \cap CB$ or $CA \cap BC$. We WLOG assume that $|AC \cap CB| > 0$.

Recall Z is constrained by six affine functions in (X, Y) . We start by choosing any set $U \subset AC \cap CB$ with positive measure. We then shrink U so that there exist sufficiently small $\epsilon > 0$ where:

- One affine constraint of Z_{\min} is tight U with margin $\epsilon > 0$ on all other constraints (i.e., if we perturb U by ϵ , this constraint is still the only one).
- On U , we have $|X| \leq \|X\|_{\infty} - \epsilon$, $|Y| \leq \|Y\|_{\infty} - \epsilon$, and $|X + Y| \leq \|X + Y\|_{\infty} - \epsilon$. This ensures our subsequent operations preserve the L^{∞} norms of X, Y, Z .
- U is either entirely used for the A -optimal matching or entirely unmatched. If matched, assume pair-sum margin is either $\geq 3\epsilon$ (A -win) or $\leq -3\epsilon$ (A -loss). If unmatched, assume that even if we improve (increase for AC , decrease for CA) the intensity by ϵ , the block remains unmatched.

We argue that the [product extension assumption](#) ensures U still has positive measure after shrinking. For (i), the joint law of (X, Y) is absolutely continuous, and so the intersections of the null faces have measure zero. This ensures that we can shrink U accordingly, should it lie on intersections of affine faces. For (ii), the level sets of $|X|, |Y|, |X + Y|$ at their respective L^{∞} -norms are null in measure, so we can shrink U to stay at least ϵ below these levels. For (iii), $U = (U \cap \{\text{matched}\}) \cup (U \cap \{\text{unmatched}\})$, so one of them must have positive measure; then there must be a positive measure further subset on which the margin is bounded away from 0, regardless of outcome.

Now we define instances I^{\pm} associated with triples $(X^{\pm}, Y^{\pm}, Z^{\pm})$, where

$$X^{\pm} = X \pm \epsilon \mathbf{1}_U, \quad Y^{\pm} = Y,$$

and Z^{\pm} is defined based on (X^{\pm}, Y) and [Equation \(1\)](#). On U , recall exactly one affine constraint is tight; with respect to this constraint, $\partial Z / \partial X \in \{0, 1/2, 1\}$. With ϵ small, the same affine face remains as the only tight constraint, so there exists $c_U \in \{0, 1/2, 1\}$ with

$$Z^{\pm} = \begin{cases} Z \pm c_U \cdot \epsilon & \text{on } U \\ Z & \text{everywhere else.} \end{cases}$$

This proves that $t(I) = (t(I^+) + t(I^-))/2$, and because U has nonzero measure, we must have $I^- \neq I^+$. As ϵ is sufficiently small, it preserves all three assumptions above. Hence voter preferences, deliberation outcomes, and $f(\cdot)$ constraints remain unchanged throughout. In particular, $I^{\pm} \in \mathcal{T}$, and $t(I)$ is not a vertex of $\text{conv}(\mathcal{T})$. \square