Due date: March 25, 2025

Problem 1: High Dimensional Cubes? Spheres! As you may have known, high-dimensional geometry is *very* weird. Many phenomena that we take for granted break down as we move into higher dimensional spaces.

(i) Consider the difference between the d-dimensional unit cube $C^d = [-1/2, 1/2]^d$ and the d-dimensional unit ball $B^d = \{x \in \mathbb{R}^d : \|x\|_2 \leqslant 1\}$. Show by example that for small d, C^d is entirely contained in B^d, but as d increases, argue that the vertices of C^d lie far outside the unit ball B^d. Part (ii) shows things are much crazier than just this!

(ii) The *Weak Law of Large Numbers* says if you have independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$, then as $n \to \infty$, the mean $Y_n = \sum_{i=1}^n X_i/n$ converges to $\mathbb{E}X_1$, and the larger n is, the more likely Y_n will be *close* (this is formally called *convergence in probability*) to $\mathbb{E}X_1$. For example if you flip a fair coin 10 times, you may find it unsurprising to get $0.4 \cdot 10 = 4$ heads. However, if you flip the coin 1000 times and observe only $0.4 \cdot 1000 = 400$ heads, you may start to think that the coin is biased, even though the fraction of heads remains 0.4.

Consider randomly sampling a point $X = (X_1, \ldots, X_n)$ in C^d by sampling each X_i uniformly from $[-1/2, 1/2]$. Then $\mathbb{E}X_i^2 = \int_{-0.5}^{0.5} x^2 \, \mathrm{d}x = 1/12$. What do you suspect

$$S_n = \sqrt{X_1^2 + \ldots + X_n^2}$$

to converge to, as $n \to \infty$? How large is this value compared to the radius of B^d? Argue (informally) that in high-dimensions, (i) the unit cube is essentially *almost* just a sphere (state its radius) and (ii) *almost all* volume of C^d lies outside of B^d.

Solution.

(i) Clearly, when $d = 1$, the line segment $[-1/2, 1/2]$ (the 1-dimensional unit cube) is entirely contained in $[-1, 1]$ (the 1-dimensional unit ball). Likewise, $[-1/2, 1/2]^2$ is entirely contained in the 2-D unit ball $\{(x, y) : \sqrt{x^2 + y^2} = 1\}$.

However, consider what happens when d is large: one of the vertices is of form $(1/2, 1/2, \ldots, 1/2)$, whose Euclidean distance to the origin is $\sqrt{(1/2)^2 + \ldots + (1/2)^2} = \sqrt{d}/2$. For $d > 4$, $\sqrt{d}/2 > 1$, and so this vertex lies outside B^d. For a drastic example, consider $d = 100$, in which case each vertex of the d-cube is of distance $\sqrt{100}/2 = 5$ from the origin — five times the radius of the unit ball!

(ii) The *Weak Law of Large Numbers* (WLLN) states that if each X_i^2 sartisfies $\mathbb{E}X_i^2 = 1/12$, then $\sum_{i=1}^n X_i^2/n$ converges (formally, *converges in probability*, a special notion of convergence which you can disregard for the purpose of this problem) to $1/12$ as $n \to \infty$. Hence $\sum_{i=1}^n X_i^2 \to n/12$, and $S_n \to \sqrt{n}/\sqrt{12}$.

The intuition (of *convergence in probability*) is that when n is *very* large, the distribution of S_n is highly concentrated around $\sqrt{n}/\sqrt{12}$. As you can see, this is an ever-growing quantity, so we obtain two results: (i) most masses of a high-dimensional cube essentially concentrates around a thin spherical shell, and (ii) the radius of this spherical shell is much larger than 1, the radius of the unit ball, as n gets large.

To prove this result rigorously, you'll need knowledge in (graduate-level) real analysis and measure theory (MATH 631), and/or measure-theoretic probability theory (MATH 641). This example is taken (informally) from Example 2.2.5 of the famous Durrett PTE5. I thought it'd be a cool part of the HW :3

Problem 2: Let X be a matrix. Describe a relationship between the eigenvalues and singular values of $A = X^T X$.

Solution. From SVD, one can write $X = U\Sigma V^T$. Since U, V are orthogonal (orthonormal) matrices and Σ diagonal,

$$(X^T X) = (U\Sigma V^T)(U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V(\Sigma^T\Sigma)V^T = V\Sigma^2 V^T. \qquad (*)$$

On the other hand, since A is symmetric positive semidefinite, its eigen-decomposition also admits $A = (V')^T\Lambda(V')$ where (V') is orthogonal and Λ diagonal. Relating this with (*), $\Lambda = \Sigma^2$, so for each i, $\lambda_i = \sigma_i^2$.

Problem 3: Let S be a set of n points in $\mathbb{R}^2$. A point $x \in \mathbb{R}^2$ is called a *center point* if any half-plane containing x contains at least $\lfloor n/3 \rfloor$ points of S. It is known that a center point always exists.

(i) Describe an $O(n^2 \log n)$-time algorithm to compute a center point of S. You can assume that the intersection of m halfplanes can be computed in $O(m \log m)$ time. (In fact, the run time can be improved to $O(n^2)$.)

(ii) Describe an $O(n)$ time randomized algorithm that compute an approximate center point $\tilde{x}$ of S with probability at least 1/2, i.e., any halfplane containing $\tilde{x}$ contains at least $n/4$ points of S.

Solution.

(i) We consider half-planes defined by lines that go through any two points in S. Some half-planes defined as such split S "nicely" into two roughly equal subsets and they impose no constraints on where a center point can be. However, if some half-plane splits S into two subsets where one subset has $< \lfloor n/3 \rfloor$ points, then we know that *any* center point *must* lie on the other half.

Therefore, a high-level algorithm can be defined as follows:

- Iterate through all half-planes defined by two points in S.
- Initialize constraints $\mathcal{H} = \{\}$.
- For each of them, check if either side has $< \lfloor n/3 \rfloor$ points. If so, add the *other* side H to $\mathcal{H}$.

– Return any point $p \in \bigcap_{H \in \mathcal{H}} H$.

Such a point is guaranteed to exist by the known result on existence of a center point.

The algorithm can be efficiently implemented using angular sweep. We first maintain the pairwise angle between any $s_1, s_2 \in S$, and for each point s, sort all other points by their relative angles to s. This step takes $\mathcal{O}(n^2 \log n)$ and allows us to efficiently collect all points on a certain side of a given half-plane, as the task now reduces to checking whether the relative angle falls within a certain 180-degree range.

There are $n(n-1)/2 = \mathcal{O}(n^2)$ pairs of points in S, leading to at most $\mathcal{O}(n^2)$ half-planes added to $\mathcal{H}$. Computing their intersection by assumption takes $\mathcal{O}(n^2 \log(n^2) = \mathcal{O}(n^2 \log n)$ time, as desired.

(ii) (From Nick Maroulis) We use ϵ-approximation. The algorithm is as follows:

– Choose a subset $R \subset S$ of a suitable constant size c;
– Compute the center point $\hat{x}$ of R by brute force or by (i);
– Return $\hat{x}$.

Any half-plane containing a true center point contains $n/3$ points of S. Our goal here is to find a point where any half-plan containing $\hat{x}$ contains at least $n/4$ points. Hence we allow the sampling to have an error of $n/12$. In other words, we need

$$\left| \frac{|H \cap R|}{|R|} - \frac{|H \cap S|}{|S|} \right| < \frac{1}{12} \qquad \text{for all half-planes } H.$$

Recall the ϵ-approximation theorem. We apply it here with $\delta = 1/2$ and $\epsilon = 1/12$. Observe there is no dependency on $n = |S|$, but rather on the VC-dimension of the collection of half-planes in $\mathbb{R}^2$, which is constantly 3. So, without going over the $\epsilon - \delta$ notations, there will be a sufficiently large *constant* size c that achieves our objective.