COMPSCI390.01: Algorithmic Foundations of Data Science
Midterm II

NAME:

Prob #	Score	Max Score
1		25
2		25
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$$X = U\Sigma V^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}.$$

Find the *second* principal component of X. Show your work.

(ii) Suppose we want to perform PCA on *any* matrix X of data with SVD $X = U\Sigma V^T$. (The matrices in this part are unrelated to those in (i).) For each of the following statements or questions, provide a *brief* response — one or two sentences suffice.

(a) [5pts] Suppose the data points lie exactly on a 2-D plane embedded in $\mathbb{R}^3$; what will PCA "discover," i.e., do you predict anything special about the PCA result?

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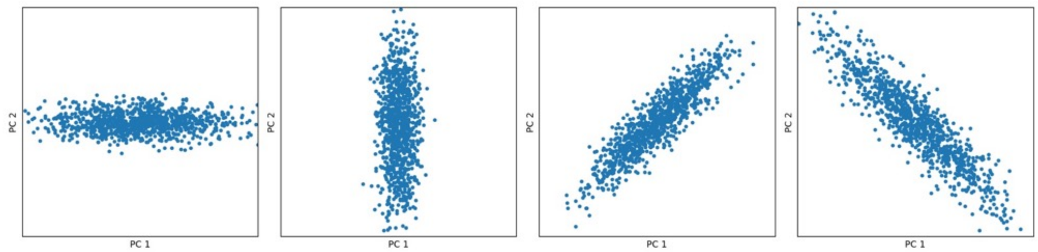

Problem 2 [25pts]: Let $X = \{x_1, \ldots, x_n\}$ be a set of n real values, and let $\varepsilon \in (0, 1)$ be a parameter. Our goal is to compute a median of X.

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Problem 3 [25pts]: Consider the undirected graph $G = (V, E)$ shown in the figure.

(i) [10pts] Write the 4×4 transition matrix P corresponding to the simple random walk on the graph. What is the stationary distribution here?

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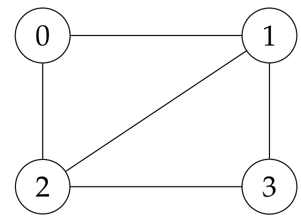

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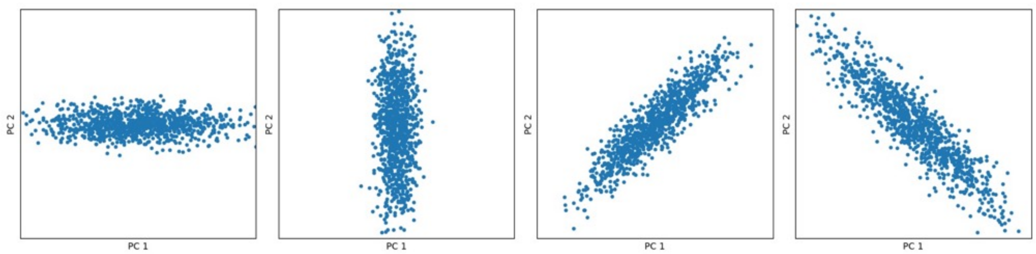

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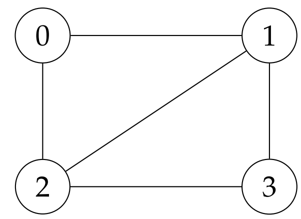

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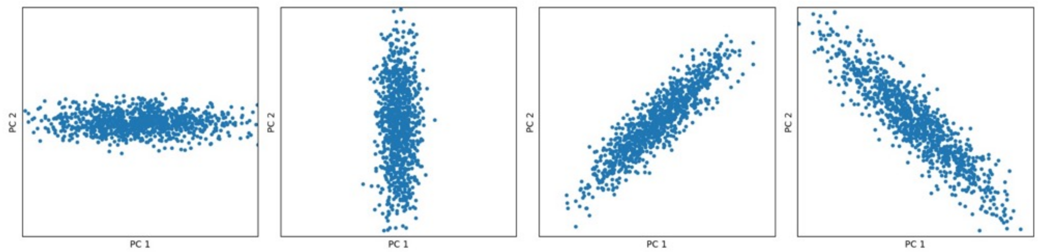

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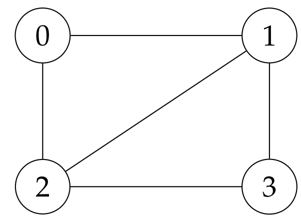

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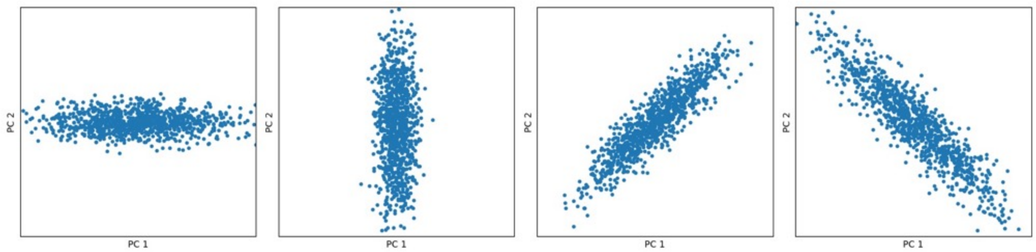

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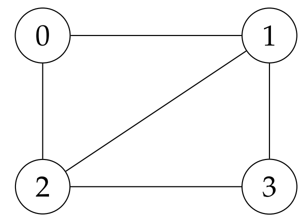

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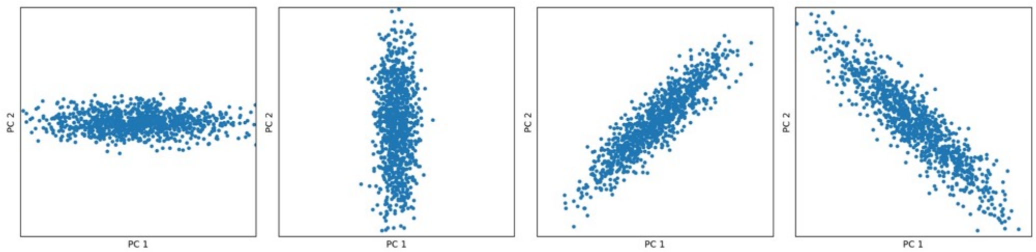

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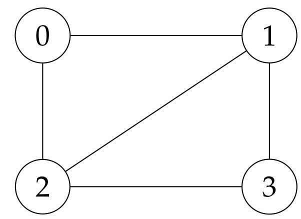

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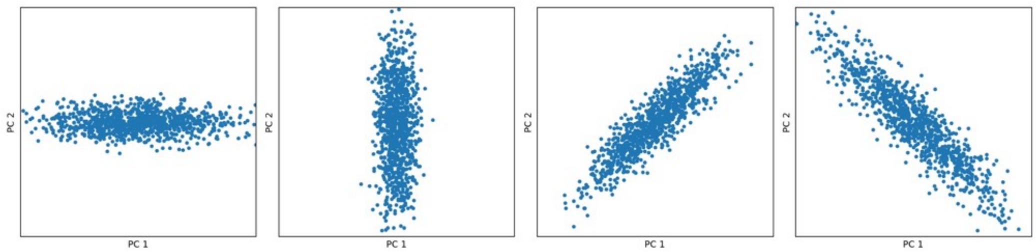

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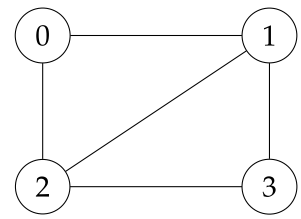

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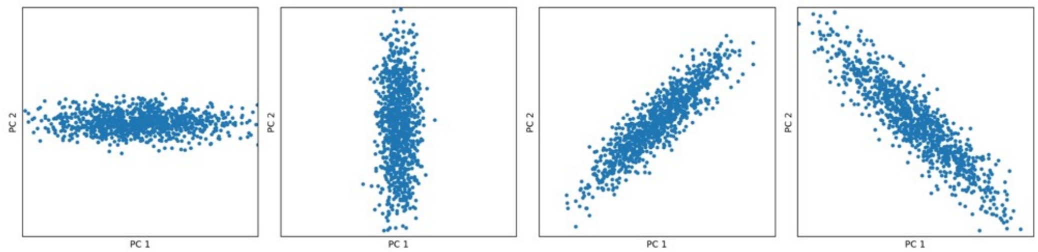

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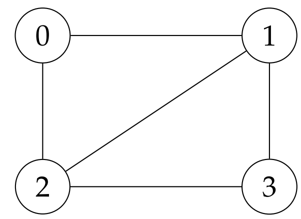

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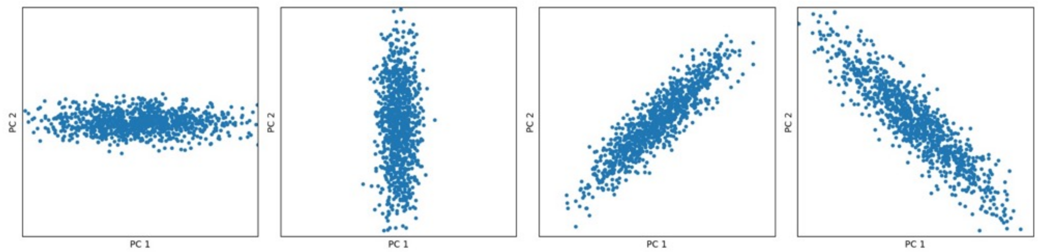

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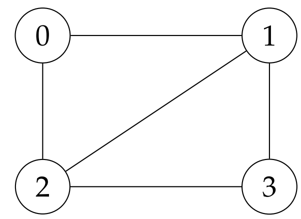

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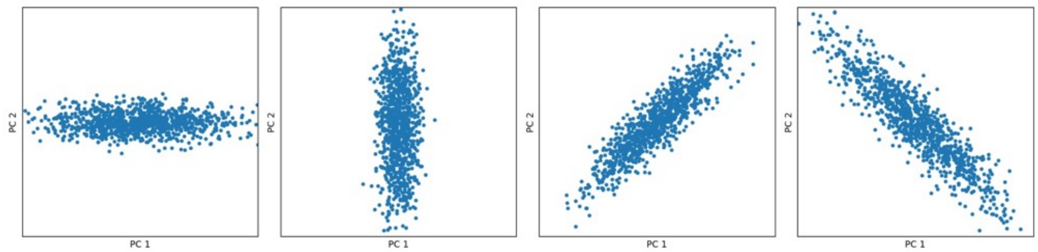

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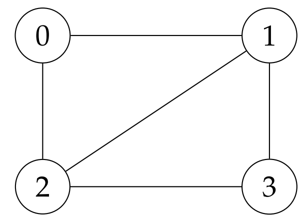

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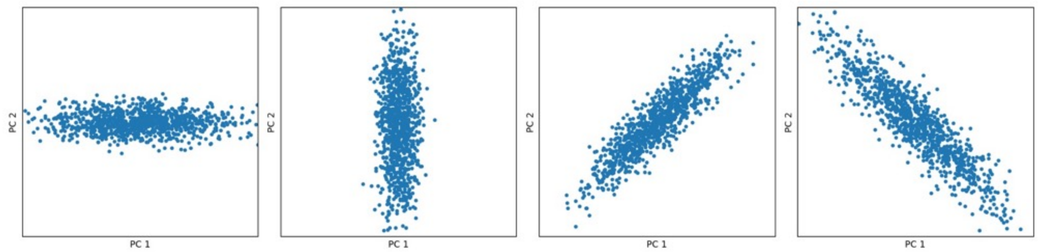


Problem 2 [25pts]: Let $X = \{x_1, \ldots, x_n\}$ be a set of n real values, and let $\varepsilon \in (0, 1)$ be a parameter. Our goal is to compute a median of X.

(i) [16pts] We choose a random sample $R \subseteq X$ of size $r = c/\varepsilon^2$, where $c > 1$ is a constant, compute the median x_R of R, and return x_R as an ε-approximate median of X. Argue that the rank of x_R in X lies in the range

$$\left[(1-\varepsilon)\frac{n}{2},\ (1+\varepsilon)\frac{n}{2}\right]$$

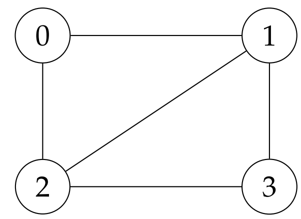
with probability at least $1/2$, provided that c is chosen sufficiently large. (Hint: use ε-approximation.)

(ii) [9pts] Suppose we receive a stream $x_1, x_2, \ldots$ of real values, and let ε be a parameter. For any $t > 0$, let $X_t = \langle x_1, \ldots, x_t \rangle$. Describe an algorithm that uses $O(1/\varepsilon^2)$ space and that at any given time $t > 0$, maintains an element x^* of X_t that is an ε-approximate median of X_t, i.e., the rank of x^* in X_t lies in the range $[(1-\varepsilon)t/2,\ (1+\varepsilon)t/2]$, with probability at least $1/2$. The algorithm may use $O(1/\varepsilon^2)$ time per update, but it cannot use more than $O(1/\varepsilon^2)$ space.

Problem 3 [25pts]: Consider the undirected graph $G = (V, E)$ shown in the figure.

(i) [10pts] Write the 4×4 transition matrix P corresponding to the simple random walk on the graph. What is the stationary distribution here?

(ii) [15pts] Suppose we want a stationary distribution where $\pi(0) = 1/2, \pi(1) = 1/4$, and $\pi(2) = \pi(3) = 1/8$. Find a transition matrix P that achieves this goal. (**Hint:** *Metropolis-Hastings.*)



Problem 4 [25pts]: Suppose there is a sensitive data set $X \subset \mathbb{R}^1$ consisting of real values, which we can only access by asking interval queries, namely, given a query interval $I = [a, b]$, return $|X \cap [a, b]|$, the number of points inside I. The system wishes to guarantee $(0.1, 0)$-differential privacy, which is implemented using a Laplace mechanism $\mathrm{Lap}(b)$ for some parameter $b > 0$, so for a query I, it returns a value $y = |X \cap I| + z$, where z is a random value drawn from the Laplace distribution $\mathrm{Lap}(b)$.

The goal is to build a histogram of X consisting of $k = 200$ buckets, by asking k queries of the form $I_j = [a_j, a_{j+1}]$, for $1 \leq j < k$, where a_j's are the histogram boundaries. Let $y_j = |X \cap I_j|$ and $Y = [y_1, \ldots, y_k]$. Let $\tilde{y}_i$ be the answer returned by the system for the query I_j, so the histogram we construct is $\tilde{Y} = [\tilde{y}_1, \ldots, \tilde{y}_k]$.

(a) [10pts] What value of b should the system use to ensure $(0.1, 0)$-differential privacy?

(b) [15pts] Give an upper bound on the error $\|\tilde{Y} - Y\|_\infty = \max_{1 \leq j \leq k} |\tilde{y}_i - y_i|$ that holds with probability at least 99% under a $(0.1, 0)$-DP Laplace mechanism. Recall that for $z \sim \mathrm{Lap}(b)$, $\Pr[z > t \cdot b] < e^{-t}$.