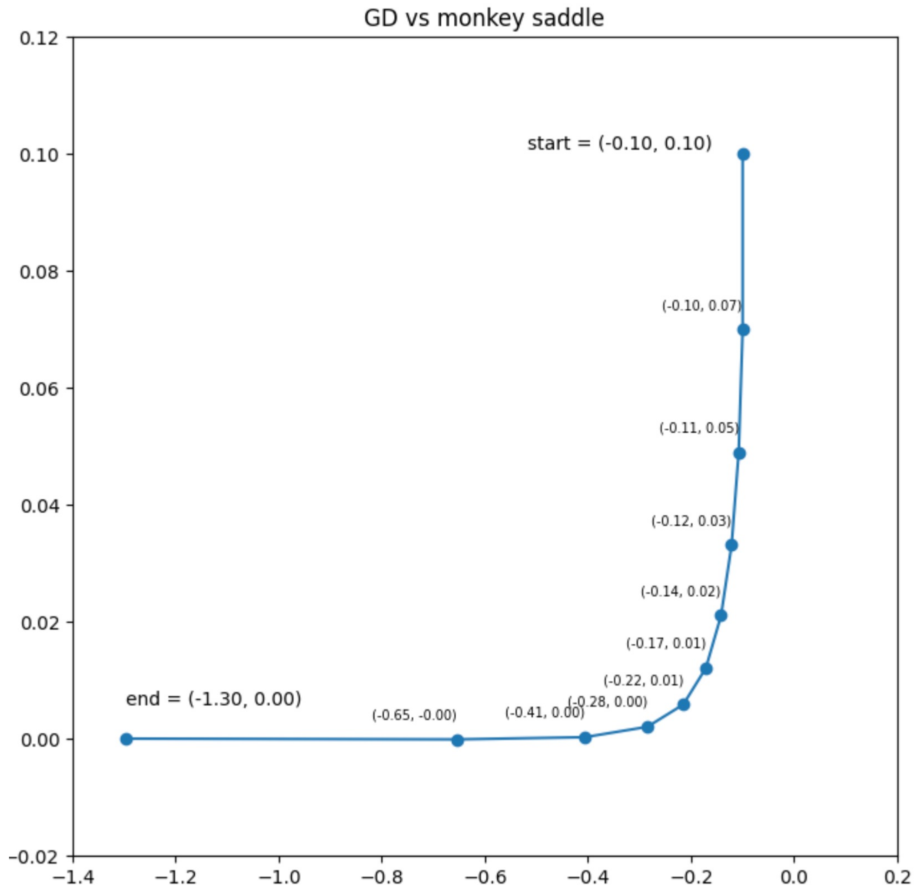
Solution to problem 1. (a) See figure below.



(b) Given δ there exists $\xi \in (0, \delta)$ where

$$g(\delta) = f(x + \delta v) = f(x) + \delta g'(0) + \frac{\delta^2}{2} g''(0) + \frac{\delta^3}{3!} g^{(3)}(0) + \frac{\delta^4}{4!} g^{(4)}(\xi)$$

Using $\nabla f(x) = 0$ as well as $v \in S$, the terms $\delta g'(0)$ and $\delta^2/2g''(0)$ vanish, so

$$g(\delta) = f(x) + \frac{\delta^3}{6} g^{(3)}(0) + \frac{\delta^4}{24} g^{(4)}(\xi).$$

WLOG assume $g^{(3)}(0) > 0$. Since $g^{(4)}$ is bounded, the last term is negligible for sufficiently small δ. More formally there exists $\delta_0 > 0$ such that if $|\delta| < \delta_0$ then $|g^{(4)}(\xi)\delta^4/24| < |g^{(3)}(0)\delta^3/6|/2$. It follows that

$$g(\delta) - f(x) = \frac{\delta^3}{6} g^{(3)}(0) + \frac{\delta^4}{24} g^{(4)}(\xi) \begin{cases} > g^{(3)}(0)\delta^3/6 - g^{(3)}(0)\delta^3/12 > 0 & \text{if } \delta \in (0, \delta_0) \\ < g^{(3)}(0)\delta^3/6 + g^{(3)}(0)\delta^3/12 < 0 & \text{if } \delta \in (-\delta_0, 0). \end{cases}$$

That is, in a sufficiently small neighborhood of x, $f(x)$ can both increase or decrease so x is neither a local minimum nor maximum, meaning it is a saddle point. The case $g^{(3)}(0) < 0$ is symmetrical.

Solution to problem 2. (a) Per the hint we decompose x as $x = \lambda u + v$ where $u \perp v$. Let $z = \langle u, a\rangle$ and $w = \langle v, a\rangle$. It follows that $z \sim \mathcal{N}(0,1)$, $w \sim \mathcal{N}(0,\sigma^2)$, and z, w are independent. Then

$$\mathbb{E}[\langle u, a\rangle^2 - \langle x, a\rangle^2]^2 = \mathbb{E}[z^2 - (\lambda z + w)^2]^2 = \mathbb{E}[z^2(1-\lambda^2) - 2\lambda z w - w^2]^2.$$

We analyze these three terms separately:

- $\mathbb{E}[z^2(1-\lambda^2)]^2 = (1-\lambda^2)^2\mathbb{E}[z^4] = 3(1-\lambda^2)^2$.
- $\mathbb{E}[2\lambda zw]^2 = 4\lambda^2\mathbb{E}[zw]^2 = 4\lambda^2\mathbb{E}[z^2]\mathbb{E}[w^2] = 4\lambda^2\|v\|^2$.
- $\mathbb{E}[w^2]^2 = \mathbb{E}[w^4] = 3\|v\|^2$.
- As for the cross terms: $\mathbb{E}[(z^2(1-\lambda^2))(2\lambda zw)]$ contains a factor $\mathbb{E}[z^3]$ which equals 0. Likewise, $\mathbb{E}[(2\lambda zw)(w^2)]$ contains $\mathbb{E}[w^3]$ which equals 0. Finally, $\mathbb{E}[(z^2(1-\lambda^2))(w^2)] = (1-\lambda^2)\mathbb{E}[z^2]\mathbb{E}[w^2] = (1-\lambda^2)\|v\|^2$.

Putting everything together we obtain the final expression,

$$\begin{aligned}\mathbb{E}[\langle u, a\rangle^2 - \langle x, a\rangle^2] &= 3(1-\lambda^2)^2 + 4\lambda^2\|v\|^2 + 3\|v\|^4 - 2(1-\lambda^2)\|v\|^2 \\ &= 3(\|x\|^2 - 1)^2 + 4(\|x\|^2 - \langle x, u\rangle^2),\end{aligned}$$

using $(1-\lambda^2) = 1 - \langle x, u\rangle^2$ and $\|v\|^2 = \|x\|^2 - \langle x, u\rangle^2$.

(b) To find first order stationary points we want the gradient

$$\nabla f(x) = 12(\|x\|^2 - 1)x - 8(x - \langle x, u\rangle u) = 4(3\|x\|^2 - 1)x - 8\langle x, u\rangle u$$

to vanish. Observe if $\langle x, u\rangle \neq 0$ then x must be a scalar multiple of u, and this forces $x = \pm u$. Otherwise, we need $x = 0$ or $3\|x\|^2 = 1$ or $\|x\|^2 = 1/3$. Thus the set of stationary points is

$$\{0\} \cup \{\pm u\} \cup \{x : \langle x, u\rangle = 0, \|x\|^2 = 1/3\}.$$

(c) Clearly, at $\pm u$ the function attains global minimum so these are true second-order stationary points.

For $x = 0$ we consider perturbing the point along u:

$$f(tu) = 3(t^2-1)^2 + 4(t^2 - t^2) = 3(t^2-1)^2$$

which clearly $< f(0) = 3$ for small t.

Finally, for x with $\langle x, u\rangle = 0$ and $\|x\|^2 = 1/3$, we consider $x(t) = x + tu$. Immediately,

$$\|x(t)\|^2 = \|x + tu\|^2 = \|x\|^2 + 0 + t^2\|u\|^2 = \frac{1}{3} + t^2$$

and $\langle x(t), u\rangle = \langle x + tu, u\rangle = t$. This gives

$$\begin{aligned}f(x(t)) &= 3(\|x(t)\|^2 - 1)^2 + 4(\|x(t)\|^2 - \langle x(t), u\rangle^2) \\ &= 3(-2/3 + t^2)^2 + \frac{4}{3}.\end{aligned}$$

Plotting this graph again yields $f(x(t)) < f(x(0))$ for sufficiently small t, so $\nabla^2 f(x)$ cannot be PSD at such x. This completes our proof.