CPS630: Homework 3

Your submission must be written up in LaTeX.

Problem 1: A parking lot attendant has mixed up n keys for n cars. The n owners arrive together. The attendant chooses a random permutation of the keys and gives each owner one key. If an owner receives his key, he leaves. The attendant repeats the process with the remaining owners and remaining keys. Use the stopping theorem to calculate the expected number of rounds after which all owners receive their key. As a first step, define X_i as the number of owners receiving their keys in the ith round. Show that

$$\mathbf{E}[X_i|X_1, X_2, \ldots, X_{i-1}] = 1$$

How will you use this to define a martingale and apply the stopping theorem?

Problem 2: Consider a bag with r red balls and g green balls. Suppose we uniformly sample n balls without replacement.

- Define a martingale for this problem analogous to the ballot theorem.

- Show that the expected number of red balls in the sample is $nr/(r+g)$.

- Use Azuma's inequality to show that the number of red balls in the sample is concentrated around the expectation. What assumption do you need to make on n relative to r and g in order for the bound to show reasonable concentration?

Problem 3: Consider a random graph G with n vertices and cn edges for constant $c > 0$. This graph is generated as follows. Let P denote the set of all $\binom{n}{2}$ pairs of vertices. Choose a pair at random, add this edge to G; and delete the pair from P. Now again choose a random pair from P and repeat, till cn edges have been added. Let X denote the number of isolated vertices in G.

- Determine upper and lower bounds on $\mathbf{E}[X]$ that are within a constant factor.

- Show that
$$\Pr\left(|X - \mathbf{E}[X]| \geq 2\lambda\sqrt{cn}\right) \leq 2e^{-\lambda^2/2}$$

Problem 4: Consider the gambler's ruin problem. A gambler starts with zero dollars; at each step he wins one dollar with probability $1/2$ and loses a dollar with probability $1/2$. The game runs till the player either wins l_1 dollars or loses l_2 dollars in total.

- Let $Z_n = \left(\sum_{i=1}^n X_i\right)^2 - n$. Show that $Z_1, Z_2, \ldots$ is a martingale.

- Let T denote the stopping time. Calculate $\mathbf{E}[Z_T]$.

- Show that the expected number of plays is $l_1 l_2$.

Problem 5: Consider a n-dimensional hypercube with $N = 2^n$ vertices. Let S denote a non-empty subset of vertices. Let x be a vertex chosen uniformly at random from the set of all vertices. Let $D(x, S)$ denote the distance between x and S, which is the minimum number of coordinates in which x and y differ over all $y \in S$.

- Provide a lower bound on $\mathbf{E}[D(x, S)]$. How does your bound depend on $|S|$?

  `Hint:` You can draw inspiration from how we lower bounded the Euclidean TSP length.

- Use Azuma's inequality to give a bound on:

$$\Pr\left[|D(x, S) - \mathbf{E}[D(x, S)]| > \lambda\right].$$

  `Hint.` Consider the martingale where we reveal one bit of x at a time. Again draw inspiration for how we derived a concentration bound for Euclidean TSP.