CPS630: Homework 4

Your submission must be written up in LaTeX.

Problem 1: Consider an irreducible finite state Markov chain whose transition probability matrix has not only each row-sum equal 1, but each column sum also equal 1. What is the stationary distribution of such a Markov chain? Prove your answer.

Problem 2: Consider an undirected graph G, and the associated Markov chain. Suppose we are at state v. Then with probability 1/2 we stay at v, and with probability 1/2 we move to a neighbor of v chosen uniformly at random. Suppose a cat and a mouse start at different vertices, and make moves according to this Markov chain. If they end up at the same vertex at the same time, the cat eats the mouse. Show that the expected time after which this happens is $O(nm^2)$ regardless of where the cat and mouse start, where n, m are the number of vertices and edges in G respectively.

Problem 3: Consider a Markov chain with $n + 1$ states $0, 1, 2, \ldots, n$. For state i, there are two transitions: To state 0 with probability 1/2 and to state $i + 1$ with probability 1/2. If $i = n$, the transition is to state n with probability 1/2 and state 0 with probability 1/2. What is the stationary distribution of this Markov chain?

Problem 4: Consider a Markov chain on n points $\{0, 1, \ldots, n - 1\}$ lying in order on a circle. At each step, the chain stays at the current state with probability 1/2 or moves to the adjacent state in the clockwise direction with probability 1/2. Find the stationary distribution, and show that $\tau(\epsilon) = O(n^2 \ln(1/\epsilon))$.

Problem 5: A Δ-coloring C of an undirected graph $G(V, E)$ is an assignment of one label (or color) to each vertex, where the color is drawn from $\{1, 2, \ldots, \Delta\}$. An edge (u, v) is improper if both end-points have the same color. Let $I(C)$ be the number of improper edges in the coloring C. Design a Markov chain on the colorings of the graph based on the Metropolis algorithm, such that in the stationary distribution, the probability of coloring C is proportional to $\lambda^{I(C)}$ for given $\lambda > 0$. Pairs of states of the chain are connected if the corresponding colorings differ in that of just one vertex.