Beginning of Feb 07, 2025

Martingales

A sequence of random variables $X_0, X_1, \cdots$ is called a **martingale** if for all i,

$$\mathbb{E}[X_i \mid X_0, \cdots, X_{i-1}] = X_{i-1}.$$

In other words, if $X_i = X_{i-1} + Y_i$ then the definition requires that $\mathbb{E}[Y_i \mid X_0, \cdots, X_{i-1}] = 0$.
For a simple example, consider a sequence of coin tosses from $\{H, T\}$. We earn one dollar for every H, and lose one for every T. Let X_i be our profit/loss at time i. Clearly, for any i, $Y_i = X_i - X_{i-1}$ takes values ± 1 with equal probabilities.

Theorem: (Wald) Optional Stopping Theorem

Given a martingale $\{X_n\}_{n \geqslant 0}$, a **stopping time** T is a random variable such that $\mathbb{P}(T = n)$ depends only on $X_0, \cdots, X_n$ but not X_{n+1} onwards. The optional stopping theorem says whenever T is finite, $\mathbb{E}X_T = X_0$. Roughly speaking, *one cannot time the market* if given finite time.

Theorem: Ballet Theorem

Suppose there are two candidates A, B for an election. Suppose A eventually has a votes, B has b votes, and $a > b$. Then, during the vote counting process, assuming votes are counted randomly,

$$\mathbb{P}(A \text{ always ahead of } B) = \frac{a-b}{a+b}.$$

Proof. Many proofs exist, for example proof by reflection or by induction. Here we use stopping time.
Let $n = a + b$. Define $S_k =$ lead of A after k steps. Clearly, $S_0 = 0$ and $S_n = a - b$. We then define the backwards variables $X_k = S_{n-k}/(n-k)$. This gives $X_0 = S_n/n = (a-b)/(a+b)$ and

$$X_{n-1} = S_1 = \begin{cases} +1 & \text{with probability } a/(a+b) \\ -1 & \text{with probability } b/(a+b). \end{cases}$$

Claim: $X_0, X_1, \cdots, X_{n-1}$ form a martingale.
To see this, let $q = S_{n-k+1}$. We know that by $n-k+1$ steps, the vote count for A is $(n-k+1+q)/2$ and likewise, the vote count for B is $(n-k+1-q)/2$. Therefore, the $(n-k+1)^{\text{th}}$ vote can take the following events:

$$(n-k+1)^{\text{th}} \text{ vote} = \begin{cases} A \text{ with probability } \dfrac{n-k+1+q}{2(n-k+1)} \Rightarrow S_{n-k} = S_{n-k+1} - 1 \\ B \text{ with probability } \dfrac{n-k+1-q}{2(n-k+1)} \Rightarrow S_{n-k} = S_{n-k+1} + 1. \end{cases}$$

Therefore,

$$\mathbb{E}[S_{n-k} \mid S_{n-k+1}] = [\text{algebra}] = S_{n-k+1} \cdot \frac{n-k}{n-k+1}$$

so dividing both sides by $n-k$ gives $\mathbb{E}[X_k \mid X_{k-1}] = X_{k-1}$. END OF PROOF OF CLAIM

Now define a stopping time T to be the minimum k such that $X_k = 0$, or $n - 1$ if no such k exists. This ensures T is finite, and the probability that A keeps its lead during the entire counting process is $\mathbb{E}X_T$. The optional stopping theorem then gives $\mathbb{E}X_T = X_0 = (a - b)/(a + b)$, as claimed. □