Beginning of 03/07/2025

Consider graph coloring. Our goal is to randomly sample a coloring from it.
Assume $c > 4\Delta$ where Δ is max degree. Clearly there are exponentially many colorings, so an exhaustive search is not feasible. Instead consider the following randomized algorithm:

- Pick $v \in V$ at random.
- Pick color $q \in \{1, \cdots, c\}$ at random.
- If feasible to assign v color q, do so; else do nothing. Repeat.

Claim 1. The stationary state is a uniform distribution over feasible colorings.

Theorem

Given a MC with transition matrix P, suppose there exists π such that

$$\pi_x P_{x,y} = \pi_y P_{y,x} \qquad \text{for all } x, y \in S, \tag{*}$$

then π is proportional to a stationary distribution.

Proof. A stationary distribution satisfies $\hat{\pi} P = \hat{\pi}$, so $\hat{\pi}_y = \sum_x \hat{\pi}_x P_{x,y}$.
Summing (*) over x gives

$$\sum_x \pi_x P_{x,y} = \sum_x \pi_y P_{y,x} = \pi_y \sum_x P_{y,x} = \pi_y$$

and the proof is complete. □

It immediately follows that $P_{x,y} = P_{y,x}$ for all x, y, then the stationary distribution is uniform. We will apply this property to showing the randomized algorithm indeed draws colorings from a uniform distribution.

Consider a pair of coloring x and y that differ by one vertex (so their states are connected by an edge). Say this is vertex v. Say it is colored as q_1, q_2, respectively, in x and y.
How to calculate $P_{x,y}$? We need to pick v in step 1 and then try to color it into q_2 in step 2. Hence $P_{x,y} = (1/n) \cdot (1/c)$. The case for $P_{y,x}$ is symmetric. Therefore the randomized algorithm corresponds to a MC whose stationary distribution is uniform.

Now for mixing time where we use $c > 4\Delta$. Consider two copies of the MC. The coupling allow both chains to pick the same $\langle v, q \rangle$ (the outcome of step 3 doesn't matter).

Theorem

$\tau(\epsilon) \leqslant \left\lceil \frac{nc}{c - 4\Delta} \cdot \log(n/\epsilon) \right\rceil$. Essentially, the time it takes for the distribution to be close to uniform is constant.

Proof. Define randomly variables d_t to be the number of vertices colored differently in both chains. Hence at time t, d_t vertices are colored differently and $n - d_t$ the same.
We want to show $\mathbb{E}[d_t]$ decreases as t increases, so the difference diminishes.
Case 0. Suppose the random vertex v picked is among the $n - d_t$ "same" group. □