Recall the definitions of Copeland-deliberation in `deliberation.pdf`. We now attempt to prove the two-hop case: we want to derive a tight upper bound of $SC(A)/SC(B)$ subject to the assumption that A is in the λ-weighted uncovered set (λ-WUS). That is, with respect to the normalized Copeland-deliberation score $f(\cdot)$, for every other B, either $f(AB) \geqslant 1-\lambda$, or there exists another C such that $f(AC) \geqslant 1-\lambda$ and $f(CB) \geqslant \lambda$. We refer to them as the easy and the hard case, respectively.

In section 1.3 of `deliberation.pdf`, we proved that if $f(AB) \geqslant r$, then

$$\frac{SC(A)}{SC(B)} \leqslant F(r) = \begin{cases} (2-r)/r - 1 = 2/r - 2 & \text{if} \quad 0 \leqslant r \leqslant 1/2 \\ (1+r)/r - 1 = 1/r & \text{if} \quad 1/2 \leqslant r \leqslant 1. \end{cases}$$

Substituting $r = 1-\lambda$ into the equation gives the closed form expression on distortion bound. In what follows, we work exclusively with the hard case, characterized by the following assumptions.

Assumption 1. *Let A, C, B be candidates and $\lambda \in [0.5, 1]$. Assume $f(AC) \geqslant 1-\lambda$ and $f(CB) \geqslant \lambda$.*

Following standard methods used in ABE18 and EC19, we consider two scenarios depending on whether $d(C,B) \geqslant d(A,B)$ or $d(C,B) < d(A,B)$. We keep notations from previous sections. In particular, we define preference sets $XY = \{v \in V : d(v,X) < d(v,Y)\}$ and $XYZ = \{v \in V : d(v,X) < d(v,Y) < d(v,Z)\}$. For each pair of candidates (including $A-B$), fix the deliberation that happen between them. Let x_{XY} denote the number of *pairs* that favor X.

1 Case (A): $d(C,B) \geqslant d(A,B)$

1.1 A Simple Bound $2 + 1/\lambda$

We now return to the general Case (1) problem, dropping assumptions on the sizes of $|BC|, |CB|$, and provide two analytic bounds that are slightly loose.

Recall that in the $C-B$ deliberation, each C-win pair gives $d(u,B) + d(v,B) \geqslant d(C,B)$ to the losing side B and there are x_{CB} such pairs total. For every other voter that also ranks C over B, we nevertheless have $d(v,B) \geqslant 1/2 \cdot d(C,B)$. We also note that the following program

$$\begin{aligned} \text{minimize} \quad & \text{score}(CB) = c + x_c \\ \text{subject to} \quad & c + b = 1 \\ & x_c + x_b = \min(c,b) \\ & (c + x_c) \geqslant \lambda(c + b + x_c + x_b) \\ & a, b, x_c, x_b \geqslant 0. \end{aligned} \tag{1}$$

finds the smallest feasible quantity score(CB) subject to $f(CB) \geqslant \lambda$. Assuming $\lambda \in [0.5, 1]$, the result is characterized by letting $c = 2\lambda/(1+\lambda), x_c = 0$, and $b = x_b = 1-c$. And when this happens, the objective value is $2\lambda/(1+\lambda)$. Thus, $f(CB) \geqslant \lambda$ implies score$(CB)/n \geqslant (2\lambda)/(1+\lambda)$ for $\lambda \in [0.5, 1]$. Putting everything together,

$$SC(B) \geqslant \sum_{(u,v)\in C-\text{win}} [d(u,B) + d(v,B)] + \sum_{\text{other voters in } CB} d(v,B) \tag{2}$$

$$\geqslant x_{CB} \cdot d(C,B) + (|CB| - x_{CB})/2 \cdot d(C,B) \tag{3}$$

$$= \text{score}(CB)/2 \cdot d(C,B) \geqslant \frac{2\lambda}{1+\lambda} \cdot \frac{n}{2} \cdot d(C,B) \tag{4}$$

$$\geqslant \frac{2\lambda}{1+\lambda} \cdot \frac{|BA|}{2} \cdot d(C,B) = \frac{\lambda}{1+\lambda} \sum_{v\in BA} d(C,B) \geqslant \frac{\lambda}{1+\lambda} \sum_{v\in BA} d(A,B) \tag{5}$$

and the result $SC(A)/SC(B) \leqslant 1 + (1+\lambda)/\lambda = 2 + 1/\lambda$ now follows from the following lemma (EC19 Lemma 3.8) and Case (1)'s assumption that $d(C,B) \geqslant d(A,B)$.

Lemma 1. *For a candidate A and a voter v, let $Q_v(A)$ be the set of candidates that v likes at most as much as A, i.e., if $X \in Q_v(A)$ then $d(v,X) \geqslant d(v,A)$. If*

$$SC(B) = \sum_{v\in V} d(v,B) \geqslant \gamma \sum_{v\in BA} \min_{C\in Q_v(A)} d(B,C),$$

we have $SC(A)/SC(B) \leqslant 1 + 1/\gamma$.

Proof. The proof is just a series of symbol pushing, along with the definition of $Q_v(A)$ and triangle inequality:

$$\begin{aligned} SC(A) = \sum_{v\in V} d(v,A) &= \sum_{v\in AB} d(v,A) + \sum_{v\in BA} d(v,A) \\ &\leqslant \sum_{v\in AB} d(v,B) + \sum_{v\in BA} [d(v,A) - d(v,B) + d(v,B)] \\ &\leqslant \sum_{v\in V} d(v,B) + \sum_{v\in BA} [d(v,A) - d(v,B)] \\ &\leqslant SC(B) + \sum_{v\in BA} \min_{C\in Q_v(A)} [d(v,C) - d(v,B)] \leqslant (1+\gamma) SC(B). \end{aligned}$$ □

1.2 Distortion Bound $2 + 1/\lambda$ Is Loose

However, this result is also not tight, and we are unsure how much slackness there is. Another analysis yields

$$\frac{SC(A)}{SC(B)} = \frac{\sum_{v\in V} d(v,A)}{SC(B)} \leqslant \frac{\sum_{v\in V} d(v,B) + d(A,B)}{SC(B)} \tag{6}$$

$$= \frac{SC(B)}{SC(B)} + \frac{\sum_{v\in V} d(A,B)}{SC(B)} \tag{7}$$

$$\leqslant 1 + \frac{\sum_{v\in V} d(B,C)}{SC(B)} \leqslant 1 + \frac{\sum_{v\in V} d(v,B) + d(v,C)}{SC(B)} \tag{8}$$

$$= 2 + \frac{SC(C)}{SC(B)}. \tag{9}$$

From the earlier LP and the argument that $SC(C)/SC(B) \leqslant 2n/\text{score}(CB) - 1$, given $f(CB) \geqslant \lambda \geqslant 0.5$, we have

$$\text{score}(CB)/n \geqslant \frac{2\lambda}{1+\lambda} \implies \frac{SC(C)}{SC(B)} \leqslant \frac{2n}{(2\lambda n)/(1+\lambda)} - 1 = \frac{1}{\lambda}.$$

Thus $SC(A)/SC(B) \leqslant 2 + SC(C)/SC(B) \leqslant 2 + 1/\lambda$.

Suppose the inequalities are tight. Then we need the following to hold simultaneously:

(1) For all v, we need $d(v,A) = d(v,B) + d(A,B)$, meaning B is co-linear with and in between v and A.

(2) We need $d(A,B) = d(B,C)$.

(3) For all v, we need $d(B,C) = d(v,B) + d(v,C)$, meaning v is co-linear with and in between B and C.

(4) The voters must maximize $SC(C)/SC(B)$. *The exact layout is not needed for this particular analysis.*

(5) The LP extremal solution is attained, so in the C-B deliberation, B wins all. In particular, $|CB| = 2\lambda/(1+\lambda)$ and $|BC| = (1-\lambda)/(1+\lambda)$.

Without even caring about the distribution of voters that maximizes $SC(C)/SC(B)$, (1) and (3) imply A, B, C and all voters lie on a line. From (2), either $A = C$ or B is the midpoint of A, C. The former cannot happen, for otherwise (1) forces all voters to be co-located with B, which will force $f(CB) = 0$ (for everyone picks B as their first choice and C cannot possibly have any share in the $B - C$ contest). Thus, we assume B is the midpoint of A, C, and (3) forces all voters to lie on the $B - C$ segment.
Now recall $f(AC) \geqslant 1 - \lambda$. Even if there are deliberations, both endpoints of the deliberation pair will be on the $B - C$ segment, so A cannot win any deliberation. Even for base votes, the only possible way to place voters who rank A over C is by placing them at the midpoint B and arbitrate in favor of A. Let $|AC| = w$, i.e., we first place w voters at B with preference BAC. As $\text{score}(AC) = |AC| = w$, we must enforce

$$f(AC) = \frac{w}{1 + \min(w, 1-w)} \geqslant 1 - \lambda,$$

which gives rise to the following:

$$w \geqslant \min\left\{\frac{2-2\lambda}{2-\lambda}, \frac{1-\lambda}{\lambda}\right\} = \begin{cases} (2-2\lambda)/(2-\lambda) & \text{if } 1/2 \leqslant \lambda \leqslant 2/3 \\ (1-\lambda)/\lambda & \text{otherwise.} \end{cases}$$

On the other hand, $w > (1-\lambda)/(1+\lambda)$. Since a voter at B definitely favors B over C, we must have $|BC| \geqslant w > (1-\lambda)/(1+\lambda)$ so (5) cannot be satisfied. Therefore the inequalities cannot be simultaneously made tight, and $SC(A)/SC(B)$ is strictly dominated by $2 + 1/\lambda$, and our current goal is to lower bound this slackness.

Based on empirical results, we suspect that it helps break the problem down into two further sub-cases, depending on the size of $|CB|$ and $|BC| = n - |CB|$. Specifically, empirical evidence suggests that the former has distortion tightly bounded by $2/\lambda$ and is only feasible if $\lambda \leqslant 2/3$, while the latter is bounded by distortion 3 and feasible for any λ. To this end, we perform per-case analysis below.

1.3 Sub-Case: $|CB| \leqslant |BC|$

In this case, we may augment Program (1) with an additional constraint that $c \leqslant b$:

$$\begin{aligned} \text{minimize} \quad & \text{score}(CB) = c + x_c \\ \text{subject to} \quad & c \leqslant b \\ & c + b = 1 \\ & x_c + x_b = \min(c, b) = c \\ & (c + x_c) \geqslant \lambda(c + b + x_c + x_b) \\ & a, b, x_c, x_b \geqslant 0. \end{aligned} \tag{10}$$

This LP is feasible when $\lambda \leqslant 2/3$ and its optimal value is $\text{score}(CB) = 2\lambda/(2-\lambda)$, attained via $c = x_c = \lambda/(2-\lambda)$, $b = 1 - c$, and $x_b = 0$. Substituting this finer lower bound into the chain inequalities in Section 1.1, we obtain

$$SC(B) \geqslant \text{score}(CB)/2 \cdot d(C,B) \geqslant \frac{2\lambda}{2-\lambda} \cdot \frac{n}{2} \cdot d(C,B) \geqslant \frac{\lambda}{2-\lambda} \sum_{v \in BA} d(A,B), \tag{11}$$

and Lemma 1 now gives

$$\frac{SC(A)}{SC(B)} \leqslant 1 + \frac{2-\lambda}{\lambda} = \frac{2}{\lambda}. \tag{12}$$

We note that this result is tight; a family of instances parametrized by λ can attain this bound; see the appendix. We also note that the LP is infeasible when $\lambda > 2/3$: indeed, if $|CB| \leqslant |BC|$, then $\text{score}(CB) \leqslant 2|CB| \leqslant 2|BC|$, and it is impossible that $f(CB) = \text{score}(CB)/[\text{score}(CB) + \text{score}(BC)] > 2/3$.

1.4 Sub-Case: $|CB| \geqslant |BC|$

TO BE FINISHED. Current empirical results suggest that all $\lambda \in [0.5, 1]$ are feasible, yet the distortion is tightly bounded by 3. Proof is currently unavailable.

2 Appendix: Known Large-Distortion Instances

INSTANCE WITH DISTORTION $2/\lambda$, ASSUMING $d(B,C) \geqslant d(A,B)$.

Let A, B, C be co-linear, with $d(A,B) = d(B,C) = 0.5$ and $d(A,C) = 1$. Partition the voters into two clusters, with preference profiles CBA and BAC as follows.

cluster	dist. to A	to B	to C	weight
CBA	1	0.5	0	$\alpha = \lambda/(2-\lambda)$
BAC	0.5	0	0.5	$\beta = (2-2\lambda)/(2-\lambda)$

Note $AC = BAC$ and $CB = CBA$. This enables the following deliberations:

edge/contest	base vote	deliberation bonus	score	normalized score
AC	β	0	β	λ
CA	α	α	2α	$1-\lambda$
CB	α	α	2α	$1-\lambda$
BC	β	0	β	λ

(Observe that α and $\beta = 1-\alpha$ are chosen so that the extreme-case deliberation outcomes coincidentally lead to tight edge $f(AC) = \lambda$ and $f(CB) = 1-\lambda$.) In this instance, $SC(A) = 1/(2-\lambda)$ and $SC(B) = (\lambda/2)/(2-\lambda)$, leading to $SC(A)/SC(B) = 2/\lambda$, as claimed.