Contents

This is an informal compilation of current major results. Important sections & paragraphs are highlighted. Please expect typos, repetitions, and other random stuff :-)

1 Copeland-Deliberation Rule on 2 Candidates

1.1 The Copeland-Deliberation Rule

In the notation of ABE18, let us revisit the 2-candidate voting problem. Let A, B be the candidates. We partition the voters into AB and BA where a voter in the former prefers A over B and vice versa in BA. The standard Copeland rule declares A as winner if $a = |AB| \geqslant |BA| = b$. Let $m = \min\{|AB|, |BA|\} = \min\{a, b\}$. Our deliberation step arbitrarily forms a maximal size-m matching $M = \{(s_j, t_j)\}_{j=1}^m \subset AB \times BA$. For each pair (s_j, t_j), we say that A wins the deliberation if

$$d(s_j, A) + d(t_j, A) \leqslant d(s_j, B) + d(t_j, B).$$

Say there are x_a pairs in P that favor A, and x_b pairs that favor B. Then our rule declares A as winner if and only if $a + x_a \geqslant b + x_b$. *Compare this against standard Copeland, which simply compares a against b.*

1.2 Copeland-Deliberation Analysis: 2 Candidates

For convenience, let $\Delta = d(A, B)$, and let M be the matching formed by deliberations. Define

$$M_A = \{(u, v) \in M : A \text{ wins}\} = \{(u, v) \in M : d(u, A) + d(v, A) \leqslant d(u, B) + d(v, B)\}$$

$$M_B = \{(u, v) \in M : B \text{ wins}\} = \{(u, v) \in M : d(u, A) + d(v, A) > d(u, B) + d(v, B)\}.$$

Observe $M_A \cap M_B = \emptyset$ and $M_A \cup M_B = M$. By definition, $|M_A| = x_a$ and $|M_B| = x_b$. Define the "remainders" $R_A = AB \backslash \{u : (u, v) \in M\}$ and $R_B = BA \backslash \{v : (u, v) \in M\}$. Immediately, one (or both) of R_A or R_B must be empty. We WLOG assume that the algorithm declares A as winner. To bound the distortion, we aim to maximize $SC(A)/SC(B)$ and so we upper bound $SC(A)$ and lower bound $SC(B)$.

STEP 1. UPPER-BOUNDING $SC(A)$. Clearly, for every voter v, by triangle inequality

$$d(v, A) \leqslant d(v, B) + \mathbf{1}[v \in BA] \cdot \Delta = \begin{cases} d(v, B) & \text{if } v \in AB \\ d(v, B) + d(B, A) & \text{if } v \in BA. \end{cases} \tag{1}$$

Using M, R to partition the voters, we obtain

$$\begin{aligned} SC(A) &= \sum_{(u,v)\in M} [d(u, A) + d(v, A)] + \sum_{v \in R} d(v, A) \\ &= \sum_{(u,v)\in M_A} [d(u, A) + d(v, A)] + \sum_{(u,v)\in M_B} [d(u, A) + d(v, A)] + \sum_{v\in R_A} d(v, A) + \sum_{v\in R_B} d(v, A). \end{aligned} \tag{2}$$

We analyze the costs term by term.

(A1) On each $(u, v) \in M_A$: As A wins the deliberation, $d(u, A) + d(v, A) \leqslant d(u, B) + d(v, B)$.

(A2) On each $(u, v) \in M_B$: Two applications of (1) give $d(u, A) + d(v, A) \leqslant d(u, B) + d(v, B) + \Delta$.

(A3) (1) is also directly applicable on the last two terms in $R_A \subset AB$ and $R_B \subset BA$.

Summing over everything,

$$SC(A) \leqslant SC(B) + (x_b + |R_b|) \cdot \Delta = SC(B) + (b - x_a) \cdot \Delta \tag{3}$$

as $x_b + |R_b| = (m - x_a) + (b - m) = b - x_a$. [Recall $m = \min(a, b)$ the matching size.]

STEP 2. LOWER-BOUNDING $SC(B)$. For any pair $(u, v) \in M_A$, two applications of the triangle inequality gives

$$\begin{cases} d(u, B) + d(v, B) \geqslant d(u, A) + d(v, A) \\ d(u, A) + d(u, B) \geqslant \Delta \\ d(v, A) + d(v, B) \geqslant \Delta \end{cases} \implies d(u, B) + d(v, B) \geqslant \Delta \tag{4}$$

With this in mind, we consider certain subsets of voters and analyze their contribution to $SC(B)$:

(B1) Each $(u, v) \in M_A$ collectively contributes Δ to $SC(B)$ by (4). There are x_a such pairs.

(B2) The remaining $a - x_a$ voters in AB each contribute at least $\Delta/2$ to $SC(B)$.

Ignoring everything else, overall

$$SC(B) \geqslant x_a \cdot \Delta + (a - x_a)\Delta/2 = (a + x_a)/2 \cdot \Delta. \tag{5}$$

Combining (3) and (5), we see that

$$\begin{aligned}\frac{SC(A)}{SC(B)} &\leqslant \frac{SC(B)+(b-x_a)\Delta}{SC(B)} \leqslant 1+\frac{(b-x_a)\cdot\Delta}{(a+x_a)/2\cdot\Delta}\\ &\leqslant 1+\frac{2(b-x_a)}{a+x_a} = \frac{a+x_a+2b-2x_a}{a+x_a}\\ &= \frac{2a+2b-a-x_a}{a+x_a} = \frac{2n}{a+x_a}-1. \end{aligned} \tag{6}$$

Theorem 1. *Suppose there are two candidates A, B and n voters. Among the electorate, suppose $a \leqslant n$ voters prefer A. Further, suppose that during the deliberation round, x_a pairs end up favoring A. Define $\text{score}(A) = a + x_a$ and likewise for $\text{score}(B)$. Then,*

$$\frac{SC(A)}{SC(B)} \leqslant 1+\frac{b-x_a}{(a+x_a)/2} = \frac{2n}{a+x_a}-1 = \frac{2n}{\text{score}(A)}-1. \tag{7}$$

Proposition 1. *Under this rule, the distortion for any 2-candidate instance is bounded by 2.*

Proof. By the theorem, it suffices to show that if A wins, then $\text{score}(A) \geqslant 2n/3$.

To prove this claim, we WLOG assume $|AB| \leqslant |BA|$. If A is the winner, then $a + x_a \geqslant b + x_b = b + (a - x_a)$ iff $2x_a \geqslant b = n - a$. But we also have $x_a \leqslant a$, so $2x_a \leqslant 2a$ and $n - a \leqslant 2a$ implies $a \geqslant n/3$. Hence

$$\text{score}(A) = a + x_a \geqslant a + \frac{n-a}{2} = \frac{n+a}{2} \geqslant \frac{n+(n/3)}{2} = \frac{2n}{3}.$$

If instead $|AB| \geqslant |BA|$, then the winner condition forces $a + x_a \geqslant b + x_b = b + (b - x_a) = 2b - x_a$, or equivalently $2x_a \geqslant 2b - a$. If $a \geqslant 2n/3$ there is nothing to show, so we assume $n/2 \leqslant a \leqslant 2n/3$. In this case, the requirement becomes $2x_a \geqslant 2(n - a) - a = 2n - 3a$. Then,

$$\text{score}(A) = a + x_a \geqslant a + \frac{2n-3a}{2} = \frac{2n-a}{2} \geqslant \frac{2n-2n/3}{2} = \frac{2n}{3}.$$ □

1.3 Comparison Against Standard Copeland

Recall that the standard Copeland rule only compares $a = |AB|$ against $b = |BA|$. It is well known that given two candidates A, B, we always have $SC(A)/SC(B) \leqslant 2n/|AB| - 1$, to which our Theorem 1 is analogous. To prove this, we note that

$$\begin{aligned}SC(A) &= \sum_{v\in AB} d(v,A) + \sum_{v\in BA} d(v,A) \leqslant \sum_{v\in AB} d(v,B) + \sum_{v\in BA} [d(v,A)+d(v,B)]\\ &= SC(B) + |BA|\cdot d(A,B),\end{aligned}$$

so $SC(A)/SC(B) \leqslant 1 + |BA| \cdot d(A, B)/SC(B)$. Like before, we note that each AB voter contributes at least $d(A, B)/2$ to $SC(B)$, so

$$\frac{SC(A)}{SC(B)} \leqslant 1+\frac{|BA|\cdot d(A,B)}{SC(B)} \leqslant 1+\frac{(n-|AB|)\cdot d(A,B)}{|AB|\cdot d(A,B)/2} = \frac{2n}{|AB|}-1.$$

An interesting connection is as follows. In standard Copeland, we naturally define a normalized score $g(AB) = |AB|/n$ so that $g(AB) + g(BA) = 1$. Likewise, as seen in the following section, we can define a normalized Copeland-deliberation score $f(AB) = \text{score}(AB)/[\text{score}(AB) + \text{score}(BA)]$. Below we show that the distortion as functions of $f(AB)$ and $g(AB)$, differ by precisely 1 on $[0, 1/2]$.

From above, we already know that if $g(AB) = |AB|/n \geqslant r$, then the distortion is bounded by

$$G(\lambda) = 2/r - 1. \tag{8}$$

On the other hand, if given $f(AB) \geqslant r$, to use Theorem 1 we need to minimize score(AB). Thus, we solve the following program

$$\begin{aligned} \text{minimize} \quad & a + x_a \\ \text{subject to} \quad & a + b = 1 \\ & x_a + x_b = \min(a, b) \\ & (a + x_a) \geqslant \lambda(a + b + x_a + x_b) \\ & a, b, x_a, x_b \geqslant 0. \end{aligned} \tag{9}$$

which equals $2r/(2-r)$ if $r \in [0, 1/2]$ and $2r/(1+r)$ if $r \in [1/2, 1]$ (note the value w.r.t. r is continuous). Applying the Theorem, we see that if $f(AB) \geqslant r$, then the distortion is bounded by

$$F(\lambda) = \begin{cases} (2-r)/r - 1 = 2/r - 2 & \text{if} \quad 0 \leqslant r \leqslant 1/2 \\ (1+r)/r - 1 = 1/r & \text{if} \quad 1/2 \leqslant r \leqslant 1. \end{cases} \tag{10}$$

Observe that G and F differ by precisely 1 on $[0, 1/2]$, and as $r \to 1$, their difference converges to 0.

2 Extending Copeland-Deliberation to General Instance

Now we consider instances with more candidates. Notation-wise, we still let $\text{score}(AB) = |AB| + x_{AB}$ to be the deliberation-augmented score of A when compared against B. We also introduce the notion of normalized score, $f(AB) := \text{score}(AB)/[\text{score}(AB) + \text{score}(BA)]$, the benefits being that $f(AB) + f(BA) = 1$ for all distinct pairs of candidates A, B.

To construct a tournament graph, we will pick two candidates, X, Y, define an *arbitrary maximal* matching between XY and YX. Then, using the rules of Copeland-deliberation as outlined in the previous section, we compute $f(XY), f(YX)$ accordingly. Now, we repeat this process for each pair of candidates X, Y until the entire tournament graph is defined. Note that the tournament graph is not unique: given X, Y, there might be various maximal matchings between XY and YX that lead to different deliberation bonuses, and each of them give rise to a value of $f(XY)$. Additionally, the matching between $X - Y$ does not depend on any other matching, even though all such matchings are bipartite matchings built on precisely the same vertices: the full electorate. As long as the f-score of each edge is induced by a maximal matching on that edge, the tournament graph remains feasible.

Standard literature defines a candidate A to be in the λ-weighted uncovered set (WUS) if for all other B, either

- (easy case) $f(AB) \geqslant 1 - \lambda$, or
- (hard case) there exists another C such that $f(AC) \geq 1 - \lambda$ and $f(CB) \geq \lambda$.

It is known that when $\lambda \in [0.5, 1)$, the λ-WUS is nonempty. As an λ-weighted uncovered candidate can reach any other candidate in at most 2 hops via one of the above cases, to perform distortion analysis on any element in the λ-WUS, it suffices to look at a 3-candidate instance.

2.1 Default Uncovered Set: $\lambda = 0.5$

We first consider $\lambda = 0.5$. Previously, Proposition 1 has established that if $f(AB) \geqslant 0.5$ then distortion $SC(A)/SC(B) \leqslant 2$. Therefore, by the definition of a 0.5-weighted uncovered set, if A is uncovered, then $SC(A)/SC(B)$ is either at most a one-hop distortion 2, or a two-hop distortion 4. That is, the distortion of A is no more than 4. Unfortunately, this bound is tight, as illustrated by the following example (and in fact, the following section shows this is the only case, up to isomorphism[?], that tightly attains a distortion 4).

COPELAND + DELIBERATION DO NOT WORK WELL. Embed all voters on $\mathbb{R}$. Let $A = 0, C = 1, B = 0.5$. Place two voters u_1, u_2 at 0.5 and one voter v_1 at 1. Then:

candidates	base votes	deliberation	boost votes	final votes	winner
(A, C)	$\{u_1, u_2\} : \{v_1\} = 2 : 1$	no need †	N/A	A wins surely	$A \succ C$
(C, B)	$\{v_1\} : \{u_1, u_2\} = 1 : 2$	(v_1, u_1)	$1 : 0$	$2 : 2$	$C \succ B$
(B, A)	$\{u_1, u_2, v_1\} : \{\} = 3 : 0$	no need †	N/A	A wins surely	$B \succ A$

† With only one matched pair possible, the Copeland inequality is satisfied regardless of how the pair is assigned; we arbitrate in favor of A.

This example shows $A \succ C$, i.e., $f(AC) \geqslant 0.5$, and likewise $f(CB) \geqslant 0.5$, though $f(AB) = 1 - f(BA) \leqslant 0.5$

[and hence this is the two-hop case].

What Went Wrong? Characterizing 2-Candidate Distortion 2

From (6) and Theorem 1, to construct a 2-distortion instance on two candidates, all non-strict inequalities must reduce to equalities. As usual we assume A is the winner.

(i) From (A1) and (B1), for each $(u, v) \in M_A$, we need $d(u, A) + d(v, A) = d(u, B) + d(v, B) = \Delta$. Adding gives

$$d(u, A) + d(u, B) + d(v, A) + d(v, B) = 2\Delta = 2d(A, B)$$

which happens if and only if both u, v lie on the line segment $\overline{AB}$. This result along with (A1) alone further require (u, v) to be symmetrical across the midpoint of $\overline{AB}$.

(ii) (A2) requires that for each $(u, v) \in M_B$, we must have $d(u, A) = d(u, B)$ and $d(v, A) = d(v, B) + \Delta$. That is, u at the midpoint of $\overline{AB}$ as mentioned above, and v at B or anywhere beyond B on the same ray (i.e. $B + \lambda \cdot \overline{BA}$ for some $\lambda \geqslant 0$.

(iii) (B2) requires the remaining AB-voters (i.e., the ones not associated with M_A) to contribute exactly $\Delta/2$ to $SC(B)$ by lying on the midpoint of $\overline{AB}$.

(iv) (A3) requires all unmatched voters in AB to be at the midpoint of $\overline{AB}$ and all unmatched voters in BA to be precisely at B or anywhere beyond B on the same ray.

(v) Finally, when lower bounding $SC(B)$ we assumed no additional cost terms appear. Hence all voters in BA not associated with M_A must be precisely at B, so this rules out the possibility of having B-voters on the ray.

A-Voters (AB)	$u \in M_A$ pair	on $\overline{AB}$ at distance $t \in [0, \Delta/2]$ from A; partner v at $\Delta - t$
	$u \in M_B$ pair	exactly at midpoint of $\overline{AB}$
	u unmatched	exactly at midpoint of $\overline{AB}$
B-Voters (BA)	$v \in M_A$ pair	symmetric counterpart of u on $\overline{AB}$ (distance $\Delta - t$ from A)
	$v \in M_B$ pair	exactly at B
	v unmatched	exactly at B

Table 1: Permissible voter locations in distortion-2 instances.

To systematically construct examples, we'll use the invariant $a + x_a = 2n/3$ as in Proposition 1. We fix a metric scale $\Delta > 0$ and embed the two candidates at $A = 0$ and $B = \Delta$. Let $|AB| = a \in [n/3, 2n/3]$ and put $b = |BA| = n - a$. This forces $x_a = 2n/3 - a$ and $x_b = \min(a, b) - x_a$. Note by doing so we guarantee that only one side has remainders (unmatched candidates). Place voters according to Table 1, where the only freedom now is that for each of the M_A pairs, we can freely choose a $t \in [0, \Delta/2]$ and place the corresponding voters symmetrically across $\Delta/2$. That is, given parameters a, t subject to $a \in [n/3, 2n/3]$ and $t \leqslant 1/2$,

- Among the AB voters, there should be $2n/3 - a$ that win the deliberation and $2a - 2n/3$ that lose. Place the former at some distance $\Delta - t$ from A (colinear with A, B), and the remaining at the midpoint of $\overline{AB}$.
- Among the BA voters, place $2n/3 - a$ voters at some distance $\Delta - t$ from B (colinear with A, B), mirroring those AB-voters, and place the remaining all at B.

The aforementioned "bad" example exploits these properties. In this particular instance, a cluster of voters is reused both as the midpoint type and the endpoint type, making both edges along the $A - C - B$ path tight, hence achieving distortion 4.

2.2 λ-Weighted Uncovered Sets

Goal. Our next goal is to analyze the distortion of a candidate A in the λ-WUS, under the (hard case) assumption. That is, when compared against an optimal candidate B and an intermediary candidate C, the following are satisfied, given some $\lambda \in (0.5, 1)$:

- $f(AC) \geqslant 1 - \lambda$, $f(CB) \geqslant \lambda$, and
- $f(BA) \geqslant \lambda$ [for we assumed the (easy case) doesn't hold, i.e., $f(AB) < 1 - \lambda$].

Our objective is to find a strong upper bound, $G(\lambda)$, on the distortion of any candidate A selected as such. Previously (10), we proved that if the "easy case" condition holds, i.e., $f(AB) \geqslant r$, then

$$\frac{SC(A)}{SC(B)} \leqslant F(r) = \begin{cases} (2-r)/r - 1 = 2/r - 2 & \text{if} \quad 0 \leqslant r \leqslant 1/2 \\ (1+r)/r - 1 = 1/r & \text{if} \quad 1/2 \leqslant r \leqslant 1. \end{cases}$$

Substituting $r = 1 - \lambda$ into the equation gives the closed form expression on distortion bound. In what follows, we work exclusively with the hard case, characterized by the following assumptions.

Assumption 1. *Let A, C, B be candidates and $\lambda \in [0.5, 1]$. Assume $f(AC) \geqslant 1 - \lambda$ and $f(CB) \geqslant \lambda$.*

Following standard methods used in ABE18 and EC19, we consider two scenarios depending on whether $d(C, B) \geqslant d(A, B)$ or $d(C, B) < d(A, B)$. We keep notations from previous sections. In particular, we define preference sets $XY = \{v \in V : d(v, X) < d(v, Y)\}$ and $XYZ = \{v \in V : d(v, X) < d(v, Y) < d(v, Z)\}$. For each pair of candidates (including $A - B$), fix the deliberation that happen between them. Let x_{XY} denote the number of *pairs* that favor X.

3 λ-WUS Case (A): $d(C,B) \geqslant d(A,B)$

3.1 [Deprecated] A Simple Bound $2 + 1/\lambda$

We now return to the general Case (1) problem, dropping assumptions on the sizes of $|BC|, |CB|$, and provide two analytic bounds that are slightly loose.

Recall that in the $C - B$ deliberation, each C-win pair gives $d(u,B) + d(v,B) \geqslant d(C,B)$ to the losing side B and there are x_{CB} such pairs total. For every other voter that also ranks C over B, we nevertheless have $d(v,B) \geqslant 1/2 \cdot d(C,B)$. We also note that the following program

$$\begin{aligned} \text{minimize} \quad & \text{score}(CB) = c + x_c \\ \text{subject to} \quad & c + b = 1 \\ & x_c + x_b = \min(c,b) \\ & (c + x_c) \geqslant \lambda(c + b + x_c + x_b) \\ & a, b, x_c, x_b \geqslant 0. \end{aligned} \tag{11}$$

finds the smallest feasible quantity score(CB) subject to $f(CB) \geqslant \lambda$. Assuming $\lambda \in [0.5, 1]$, the result is characterized by letting $c = 2\lambda/(1+\lambda), x_c = 0$, and $b = x_b = 1 - c$. And when this happens, the objective value is $2\lambda/(1+\lambda)$. Thus, $f(CB) \geqslant \lambda$ implies score$(CB)/n \geqslant (2\lambda)/(1+\lambda)$ for $\lambda \in [0.5, 1]$. Putting everything together,

$$SC(B) \geqslant \sum_{(u,v) \in C-\text{win}} [d(u,B) + d(v,B)] + \sum_{\text{other voters in } CB} d(v,B) \tag{12}$$

$$\geqslant x_{CB} \cdot d(C,B) + (|CB| - x_{CB})/2 \cdot d(C,B) \tag{13}$$

$$= \text{score}(CB)/2 \cdot d(C,B) \geqslant \frac{2\lambda}{1+\lambda} \cdot \frac{n}{2} \cdot d(C,B) \tag{14}$$

$$\geqslant \frac{2\lambda}{1+\lambda} \cdot \frac{|BA|}{2} \cdot d(C,B) = \frac{\lambda}{1+\lambda} \sum_{v \in BA} d(C,B) \geqslant \frac{\lambda}{1+\lambda} \sum_{v \in BA} d(A,B) \tag{15}$$

and the result $SC(A)/SC(B) \leqslant 1 + (1+\lambda)/\lambda = 2 + 1/\lambda$ now follows from the following lemma (EC19 Lemma 3.8) and Case (1)'s assumption that $d(C,B) \geqslant d(A,B)$.

Lemma 1. *For a candidate A and a voter v, let $Q_v(A)$ be the set of candidates that v likes at most as much as A, i.e., if $X \in Q_v(A)$ then $d(v,X) \geqslant d(v,A)$. If*

$$SC(B) = \sum_{v \in V} d(v,B) \geqslant \gamma \sum_{v \in BA} \min_{C \in Q_v(A)} d(B,C),$$

we have $SC(A)/SC(B) \leqslant 1 + 1/\gamma$.

Proof. The proof is just a series of symbol pushing, along with the definition of $Q_v(A)$ and triangle inequality:

$$\begin{aligned} SC(A) = \sum_{v \in V} d(v,A) &= \sum_{v \in AB} d(v,A) + \sum_{v \in BA} d(v,A) \\ &\leqslant \sum_{v \in AB} d(v,B) + \sum_{v \in BA} [d(v,A) - d(v,B) + d(v,B)] \\ &\leqslant \sum_{v \in V} d(v,B) + \sum_{v \in BA} [d(v,A) - d(v,B)] \\ &\leqslant SC(B) + \sum_{v \in BA} \min_{C \in Q_v(A)} [d(v,C) - d(v,B)] \leqslant (1+\gamma) SC(B). \end{aligned}$$

□