1 Domination Graphs with Matching Certificates

1.1 High-Level Walkthrough

So far, every known deterministic rule with optimal worse-case distortion (of 3) relies, either explicitly or implicitly, on the *domination graph*. The common theme is to show that for some winner A, the corresponding domination graph G_A admits a perfect matching, which then implies that the distortion of A is no more than 3. Intuitively we knew (or hope) that pairwise deliberation may provide additional information beyond what ordinal rankings can provide. So this naturally brings out the following question:

> **Deliberation & domination graphs.** How does pairwise deliberation interact with matching-based, optimal deterministic rules? Specifically, what is a "natural" notion to incorporate deliberation into domination graphs?

To this end I considered the following "upgrade" to the domination graph of a candidate A:

- Original: an ordinal edge $i \to j$ that says voter i ranks candidate A at least as high as $\text{top}(j)$.

- New: retain only those ordinal edges that are additionally certified by a two-person deliberation between voters (i, j) on candidates $(A, \text{top}(j))$.

Formally, fix a profile and an unknown metric d and fix a candidate A. Let G_A be the corresponding (standard, ordinal) **domination graph**, bipartite with left and right copies of voters, where an edge $i \to j$ is present if voter i ranks A at least as high as $\text{top}(j)$. We define the **deliberation-refined subgraph** $H_A \subset G_A$ by keeping an edge $i \to j$ only if the deliberation between (i, j) comparing candidates $(A, \text{top}(j))$ weakly prefers A, i.e.,

$$d(i, A) + d(j, A) \leqslant d(i, \text{top}(j)) + d(j, \text{top}(j)).$$

The hoped-for certificate is: find a candidate A such that H_A admits a perfect matching, then run algebra. If this worked, each certified edge would additionally encode two-voter cardinal inequalities, so one could hope to re-run the matching argument and potentially improve the constant below 3. Pushing through triangle inequality, it is not difficult to show the following result, which shows that deliberation can reduce distortion by 1 (much alike 4.236 vs. 3.236 for $\lambda = 0.618$ uncovered set).

Theorem 1.1. *If for some candidate A, H_A has a perfect matching, then the distortion of A is at most* 2.

However, there are many major setbacks, and I have not found a fix once and for all. As of writing this summary, I tend to believe that this direction (augmenting along domination graph) does not have much potential.

- First and foremost, unlike G-graphs, **there need not exist a candidate A whose H-graph admits a perfect matching** (Theorem 1.2).

- **Even if $S = \{A \in V : H_A \text{ has a perfect matching}\}$ is non-empty, any well-defined deterministic rule selecting a winner from this set inevitably has supremum distortion** 3 (Theorem 1.3). This is proven by designing a class of instances whose S are singletons, so a deterministic selector has only one choice, but this candidate in S can be parameterized to have distortion $> 3 - \epsilon$ for any $\epsilon > 0$.

- A compromise I sought was a weaker notion: **select a candidate A whose H-graph is *close* to admitting a perfect matching**, if not already.

  We consider the "edit distance" of H_A to a perfectly matchable bipartite graph by considering both ordinal and deliberation information: most costly (2) when we wish to add an edge that is absent from both G_A and H_A, and cheapest (0) when the edge is already present in both. Otherwise the edge must have been present in G_A but not in H_A, and this costs 1. It is more convenient to describe this exact idea using a weighted bipartite graph. For a fixed candidate A, we define a complete bipartite graph on (V_ℓ, V_r) with edge weights

  - (already good) $w_A(i, j) = 0$ if $(i, j) \in H_A$,
  - (need to fix deliberation) $w_A(i, j) = 1$ if $(i, j) \in G_A \backslash H_A$, and

- (need to fix ordinal and deliberation) $w_A(i,j) = 2$ if $(i,j) \notin G_A$.

Then $\rho(A)$, candidate A's distance to the "nearest" perfectly matchable graph, can be expressed as

$$\rho(A) = \min\{\text{total weight of a perfect matching } (V_\ell, V_r) \text{ under } w_A\}.$$

This approach fixes the previous problems and has distortion at most 7/3 when $|V| = 2$. (Theorems 1.4 to 1.6). In comparison, domination-graph-based, ordinal-information-only algorithms have distortion 3 even when $|V| = 2$. **However, for general instances this rule still has supremum distortion** 3 (Theorem 1.7), and I currently believe the obstacle is structural (inherent to the rule) rather than just a bad choice of ρ.

1.2 Detailed Theorem Statements

Theorem 1.2. *There exists a family of instances in which* H_A *has no perfect matching for any candidate* A. *Furthermore, these instances can parameterized to have distortion* $\Omega(n)$.
    This claim is proven by the following construction. Let L be large. Consider a star metric with a center and n leaves labeled $1, \ldots, n$, *such that for the first* $n-1$ *branches, center-leaf distances are* 1, *and for the last one, L. For each* $k \in [n]$, *place one voter* v_k *and one candidate* c_k *right at leaf k.*

Theorem 1.3. *For all* $\epsilon > 0$, *there exists an instance with two candidates* A, B *satisfying the following:*

- G_A *admits a perfect matching, while* H_A, G_B, *and* H_B *don't, and*

- $3 > SC(A)/SC(B) > 3 - \epsilon$.

    This claim is proven by the following construction. Consider an instance on $\mathbb{R}$ *where candidate A is located at* 0 *and B at* 1. *Place* $(k+1)$ *voters at* $1/2 - \delta$, *and k voters at B = 1. Then let* $k \to \infty$.

Theorem 1.4. *By selecting the minimizer of* $\arg\min_{v \in V} \rho(v)$ *as winner, all instances considered in the proofs of Theorems 1.1 to 1.3 result in distortion* $\leqslant 2$.

Theorem 1.5. *The counterexample considered in the proof of Theorem 1.3 can be parameterized to have distortion up to* 7/3 *under the rule of selecting* $\arg\min_{v \in V} \rho(v)$.

Theorem 1.6. *The rule selecting* $\arg\min_{v \in V} \rho(v)$ *has distortion at most* 7/3 *when there are only two candidates.*

Theorem 1.7. *For all* $\epsilon > 0$, *there exists an instance in which the* ρ-*minimizer has distortion* $> 3 - \epsilon$.
    This claim is proven by another variant of the star metric. Let L, k *be given. Let the metric be a star with center* o, *with short leaves* $s_1, \ldots, s_{3k}$ *at distance* $d(o, s_i) = 1$, *and one far leaf* f *with* $d(o, f) = L$. *For each short leaf* s_i, *place one candidate and one voter. For the far leaf, place one candidate but* $k + 1$ *voters. It can then be shown that short candidates have* ρ-*value* $4k - 1$ *whereas the far candidates have* ρ-*value* $4k - 2$.