1 Matching Uncovered Set + Deliberation

1.1 A More Relaxed Notion

Another notation to consider is the **Matching Uncovered Set**, a weaker notion of domination graphs. Specifically, given candidates A, B, (the classic definition is that) we define a voter-voter bipartite graph $G(A, B)$, in which an edge (u, v) exists if there exists a candidate C such that:

(1) (left) Voter u prefers C over B, and

(2) (right) Voter v prefers A over C.

Then we select a candidate A such that for all B, the graph $G(A, B)$ has a perfect matching. With deliberation, we define $H(A, B)$ by replacing (2) and keeping (1) unchanged, and still select a candidate A such that $H(A, \cdot)$ always has a perfect matching: an edge (u, v) exists if

(1) (left) Voter u prefers C over B, and

(2') (right deliberation) The voter pair (u, v) favors A in the (A, C) deliberation.

It is easy to prove the following results:

Theorem 1.1. *If there exists a candidate A such that $H(A, B)$ has a perfect matching for each $B \neq A$, then the distortion of A is at most 2.*

Theorem 1.2. *If for some candidate A, the deliberation-augmented domination graph H_A admits a perfect matching as in* ??*, then $H(A, B)$ admits a perfect matching for each B.*

However, even with this weaker notation of matching uncovered set, there need not be a candidate A satisfying the conditions of Theorem 1.1.

Theorem 1.3. *There exists an instance in which every candidate fails to satisfy the conditions of Theorem 1.1.*

Theorem 1.4. *For every $\epsilon > 0$, there exists a metric instance with two candidates A, B such that:*

- *The (standard) matching-uncovered-set graph $G(A, B)$ admits a perfect matching, but $G(B, A)$ does not.*

- *Neither deliberation graph admits a perfect matching: $H(A, B)$ and $H(B, A)$ both fail to admit a perfect matching.*

- *B is socially optimal and $3 > SC(A)/SC(B) > 3 - \epsilon$.*

Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Work with $\mathbb{R}$ and $d(x, y) = |x - y|$. Consider an instance with 2 candidates and 3 voters, as follows: $A = 0, B = 3$, and $v_1 = 0, v_2 = 2, v_3 = 2$. Then v_1 ranks A over B, whereas v_2, v_3 rank B over A. We show that neither candidate satisfies the said requirement. In particular, neither $H(A, B)$ nor $H(B, A)$ admits a perfect matching.

First consider $H(A, B)$. For a left voter $u \in \{v_2, v_3\}$, their top choice is already B, so to satisfy (1), the witness C must be $C = B$. Thus, for left vertex $u \in \{v_2, v_3\}$, an edge (u, v) exists iff the deliberation between (u, v) comparing (A, B) weakly prefers A. It is easy to see that the only viable choice is $v = v_1$ on the right. In other words, each of the left v_2, v_3 can only be matched against the right v_1, so no perfect matching is possible.

Similarly, in $H(B, A)$, for the left voter v_1, A is top, so (1) implies that the witness C must be $C = A$. Then, an out-edge from left v_1 exists iff the deliberation between (v_1, v) comparing (B, A) weakly prefers B. This is violated by every voter, so left v_1 has no neighbors, and $H(B, A)$ has no perfect matching.

$\square$

Proof of Theorem 1.4. The construction and argument is essentially identical to that of **??**. We provide a thorough statement for the sake of completeness. We work on $\mathbb{R}$. Let k be large and $\delta > 0$ be small. Consider the instance where candidate A is placed at 0, B at 1, $(k+1)$ voters at $1/2 - \delta$ (denoted $\mathcal{L}$), and k voters at 1 (denoted $\mathcal{R}$). Then as $\delta \downarrow 0$ and $k \uparrow \infty$, the distortion $SC(A)/SC(B)$ converges to 3.

First, we show that $G(A, B)$ has a perfect matching. If $u \in \mathcal{L}$ then u ranks A over B. Choose the witness $C = A$. Condition (ii) becomes "v ranks A weakly above A," which always holds. Therefore every left vertex in $\mathcal{L}$ is connected to every right vertex. On the other hand, if $u \in \mathcal{R}$, then u strictly ranks B over A, so the only possible witness is $C = B$. Condition (ii) becomes "v ranks A over B," which holds iff $v \in \mathcal{L}$. So every left vertex in $\mathcal{R}$ connects to the right vertices in $\mathcal{L}$. Since $|\mathcal{L}| \geqslant |\mathcal{R}|$, $G(A, B)$ has a perfect matching.

We now claim $G(B, A)$ has no perfect matching. By symmetry, left $\mathcal{R}$ connects to all rights (taking $C = B$), whereas left L connects to only right $\mathcal{R}$ (take $C = A$, require "v ranks B over A," which happens iff $v \in \mathcal{R}$). Since $|\mathcal{R}| < |\mathcal{L}|$ there is no way to ensure all left-$\mathcal{L}$ vertices are matched. So $G(B, A)$ has no perfect matching.

Now we look at the H graphs. First, $H(A, B)$. As before, left $\mathcal{L}$ has edges to every right vertex via witness $C = A$, as deliberation between (A, A) is a tie so it weakly favors A. For a left voter $u \in \mathcal{R}$, the only possible witness is $C = B$. Then an edge (u, v) exists iff the pair (u, v) weakly favors A over B in deliberation. It is easy to see this is impossible, as u is a "die-hard" fan of B ($d(u, B) = 0$), which can only be countered by a "die-hard" fan of A, but this type of voter doesn't exist in this instance. So $H(A, B)$ has no perfect matching.

For $H(B, A)$, left $\mathcal{R}$ still connects to every right vertex via witness $C = B$ as deliberation between (B, B) is a tie. For a left voter $u \in \mathcal{L}$, the only possible witness is $C = A$. A pair (u, v) exists if the pair weakly prefers B over A in deliberation, and this happens iff v belongs to the right $\mathcal{R}$. But $|\mathcal{L}| > |\mathcal{R}|$, so no perfect matching exists in $H(B, A)$. □

1.2 What Went Wrong?

Conceptually, I think this is due to **discontinuity** of the objective (existence of a perfect matching) and the unerlying structure (bipartite graphs). **In our previous project, the rules were continuous w.r.t. perturbations.** Recall in our paper, we were able to continuously adjust quantities such as $f(AC)$ by shifting the marginal variables like $X_t(v) = X_0(v) - t$. When there were ties/jump discontinuities, we overcame this problem by arbitrating the ties. **The perfect matching based methods are not continuous in this regard,** as they are integral ("does there exist a matching or no?"). Small purturbations can essentially ruin a perfect matching and delete edges in a way that creates major imbalances on Hall conditions (i.e., the in/out neighbor of a set is far smaller, so no perfect matching can exist), see below. I think this is the key cause of why it is so hard to optimize in the worst case.

Additionally, changing (2) to (2') also doesn't sound canonical. An edge in the standard MUS $(u, v) \in G(A, B)$ is inherently stating a witness condition on a three-candidate chain $A - C - B$, that u is responsible for certifying "there exists a C that beats B for u," and v is responsible for certifying A beats that C for v." The whole point is that these two pieces of evidence come from different voters without interfering what each other thinks.

In the definition of $H(A, B)$, however, we lose this independence. By replacing (2) with (2'), we drag the left voter u (who was only supposed to witness $C > B$) into the A vs. C certificate itself. In some sense, u is allowed to "contaminate" the A vs. C comparison, and because of this, **an adversary can select instances where the voters C who can witness (1) are exactly the ones who force (2') to fail for almost every partner v.** This structural weakness was consistently exploited by Theorem 1.3 and in many examples in **??**. The construction in Theorem 1.3 contains "die-hard" fans who are co-located with candidate B. For any such left-side voter u, condition (1) forces $C = B$, but then for any other v, condition (2') can never hold. **Combinatorially, this creates vertices with small neighborhoods in $H(A, B)$ which prevents perfect matching from happening,** tying back to my first argument on discontinuity (as a perturbation by δ in Theorem 1.3 instantly recovers all edges needed for perfect matchings).

(That said, I believe the following notion also doesn't work. Define $G(A, B)$ via (1) and (2), and define $H(A, B)$ via (1), (2), and additionally edge (i, j) needs to satisfy that the deliberation between (i, j) over (A, B) favors A. Pick the candidate A who is "closest" to having perfect matching in $H(A, \cdot)$ in the sense of ρ defined in **??**. More likely than not, I think need a new perspective.)