1 $(\alpha - \beta)$ Unblanketed Sets

If we are satisfied with the deliberative overhead of each voter participating in $\binom{m}{2} = O(m^2)$ deliberations (as in our previous paper), then I *think* there is some potential in analyzing refinements of λ-WUS, such as the unblanketed set.

Specifically, the idea of always using optimal matchings to make constraints slack carry over. With $f(XX) = 1$ for all X (i.e., allowing duplicates, which allowed us to skip case 1 of WUS), essentially, we are simply modeling the following four-candidate scenario:

$$\text{maximize} \ \frac{SC(A)}{SC(B)} \qquad \text{subject to} \ \begin{cases} f(AD) \geqslant \beta, & f(DC) \geqslant 1-\beta \\ f(AC) \geqslant 1-\alpha, & f(CB) \geqslant \alpha. \end{cases}$$

Theorem 1.1. *Let λ^*, w^* be as in the previous paper. We showed that the λ^*-WUS (with deliberation weight w^*) tightly achieves a worst-case distortion of* 3*.*

Keep λ^*, w^* fixed, and let $\beta \in (1/2, \lambda^*]$ (this follows the parameter requirements of the unblanketed set). Unfortunately, for any such β, the (λ^*, β)-unblanketed set with deliberation weight w^* still has distortion 3*. In particular, the triangular instance in Example 6.4 can be directly extended to include a candidate D to meet the constraints.*

In other words, $(\lambda^*, \cdot)$-unblanketed sets do not upgrade λ^*-WUS. Note that this does not rule out the existence of *some* (α, β)-set from beating 3 yet, and the lower bound example below does not extend to other parameters either.

Proof. Consider a four-candidate instance defined as follows:

$$d(A,B) = d(B,C) = d(A,C) = 2, \qquad d(A,D) = 2, d(B,D) = 1, d(C,D) = 2.$$

Recall $AC_{\min} = 0.25$, $AC_{\max} = CB_{\min} = 0.50$, and $CB_{\max} = 0.75$. Define three voter clusters, named after their ordinal preferences on candidates:

Cluster name	Mass	$d(v,A)$	$d(v,B)$	$d(v,C)$	$d(v,D)$
$ADCB$	0.25	1	1	1	1
$CBAD$	0.50	3	3	1	3
$DBAC$	0.25	2	1	2	0

We claim this instance satisfies $f(AC) \geqslant 1 - \lambda^*$, $f(CB) \geqslant \lambda^*$, $f(AD) = \lambda^* \geqslant \beta$, and $f(DC) = 1/2 \geqslant 1 - \beta$.

First, for $f(AC)$, observe $|AC| = 0.5 = AC_{\max}$ so $f(AC) \geqslant 1 - \lambda^*$ regardless of deliberation outcomes; likewise, $|CB| = 0.75 = CB_{\max}$ so $f(CB) \geqslant \lambda^*$ automatically. For (A,D), as $|AD| = 0.75 = CB_{\max}$, we also know $f(AD) \geqslant \lambda^* \geqslant \beta$. Finally, for (D,C), we have $|DC| = 0.50$. For deliberation, $ADCB$ vs. $CBAD$ results in D-losses, but $DBAC$ vs. $CBAD$ can be counted as D-wins via tiebreaking, and this results in $f(DC) = (0.5 + w^* \cdot 0.25)/(1 + w^* \cdot 0.5) = 1/2 \geqslant 1 - \beta$. This completes the proof. ◻