Metric Distortion of Small-group Deliberation[†]

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Abstract

We consider models for social choice where voters rank a set of choices (or alternatives) by deliberating in small groups of size at most k, and these outcomes are aggregated by a social choice rule to find the winning alternative. We ground these models in the metric distortion framework, where the voters and alternatives are embedded in a latent metric space, with closer alternative being more desirable for a voter. We posit that the outcome of a small-group interaction optimally uses the voters' collective knowledge of the metric, either deterministically or probabilistically.

We characterize the distortion of our deliberation models for small k, showing that groups of size $k = 3$ suffice to drive the distortion bound below the deterministic metric distortion lower bound of 3, and groups of size 4 suffice to break the randomized lower bound of 2.11. We also show nearly tight asymptotic distortion bounds in the group size, showing that for any constant $\varepsilon > 0$, achieving a distortion of $1 + \varepsilon$ needs group size that only depends on $1/\varepsilon$, and not the number of alternatives. We obtain these results via formulating a basic optimization problem in small deviations of the sum of $i.i.d.$ random variables, which we solve to global optimality via non-convex optimization. The resulting bounds may be of independent interest in probability theory.

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1 Introduction

Civic discourse is the cornerstone of modern democracy. A key challenge in modern social choice theory is how to leverage the power of discourse to improve societal decision making compared to simply voting on issues. Unfortunately, there are several roadblocks to this ideal. Much of the current online discourse tends to become acrimonious, increasing polarization rather than consensus. Despite some notable successes [IL18, Fis91], it is relatively cumbersome to organize large groups for agenda-driven discussions, and even when such discussions occur, they rarely result in meaningful opinion changes, leading to the same outcomes as direct voting.

A promising approach emerging in social choice is small-group online forums, which are moderated and agenda-driven. Platforms such as the Online Deliberation Platform [sta], vTaiwan [vTa], Mismatch [mis], and Myjunto [myj] have experimented with this approach. Recent work has shown the potential of AI mediation in these settings, including facilitating human interaction via summarization and rephrasing of diverse viewpoints [ABB+23, KEO+20, sta, BCS+24, AFE+24].

Our goal is to model the outcome of such small group deliberations and analyze whether aggregating their outcomes is capable of finding a better societal alternative than would be found without such interaction. Our main result is that deliberation is strictly helpful even with small groups.

1.1 Metric Distortion and Deliberation

We use the metric distortion framework [ABE+18] to analyze the quality of the deliberation. We assume there is one issue with several potential choices (called alternatives or candidates) that the voters are deliberating over. These voters and alternatives are embedded in a latent metric space. Voters reveal ordinal preferences (that is, a ranking) over alternatives that are consistent with the latent distances, where if an alternative is closer to the voter, it is more similar to the voter's opinion, hence ranked higher by the voter. We assume the set of alternatives is common to all voters, and a voter ranks all alternatives.

A social choice rule such as the Borda score or Copeland rule takes the rankings for all voters as input, and outputs an aggregate alternative. Given an alternative, its 1-median cost is the sum of the distances of all voters to this alternative. Consider all metrics consistent with voter rankings. The distortion is the worst case for all rankings and all metrics consistent with these rankings, of the ratio of the 1-median cost of the alternative output by the social choice rule (that only knows the ordinal rankings) to the cost of the best alternative had the consistent metric been known. A lower distortion implies a better social choice rule, whose outcome is closer to the true 1-median.

It is known that in the absence of voter interaction, any deterministic voting rule that uses only the ordinal preferences of the voters has distortion at least 3 [ABE+18] and any randomized rule has distortion at least 2.11 [AP17, CR22] The key question we ask in this paper is:

Question 1. Can small group deliberation improve these bounds?

To study this question formally, we assume that voters deliberate in a small group and collectively reveal an ordinal preference (or ranking) that is consistent with their local view of the latent metric space. We consider several models of what "consistent" means, but a simple, canonical model is that the voters in the group could output the alternative that minimizes the sum of their distances to it (that is, their 1-median alternative). The 1-median intuitively captures persuasion – consider two alternatives and two voters. If the first voter is very close to the first alternative, then they care strongly about that alternative. If the second voter is equidistant from both alternatives, they are indifferent. In this setting, the first voter can persuade the second to collectively output the first alternative, which is indeed the 1-median. Even in the 1-median setting, Question 1 is far from trivial. Indeed, the example below from [FGMS17, CMP24] shows that even if all possible groups of voters of constant size output their 1-median alternative, and these are aggregated by any social choice rule, the metric distortion remains 3.

Example 1. Assume the group size is k, a constant. For some $\delta > 0$, there are n voters, who are all at distance $1 + \delta$ from a common candidate c. For each subset S of voters of size k, there is a candidate c_S that is at distance 1 from the members of S, and distance 3 from all other voters. Each group S outputs c_S when it deliberates, since this minimizes their 1-median cost – indeed, this alternative is preferred by all voters in S, leading to a trivial deliberation. Now any social choice method that aggregates the outcomes of these deliberations into a final winning alternative must produce one of these c_S as the final alternative. However, the average distance of all voters to c is $1 + \delta$, while for any c_S, the average distance is close to 3. Therefore, the distortion is at least 3.

To get around this lower bound, the group size needs to be super-constant, and indeed, the recent work of [CMP24] shows that the 1-median of a random sample of voters of size logarithmic in the number of candidates suffices to break the lower bound. However, such large groups lead to suboptimal group dynamics such as conformity, informational influence, and stereotyping of participants, which have been widely studied in psychology [Asc55, DG55] and political science [PNC06]. In fact, the celebrated Asch experiments [Asc55] show that conformity begins to kick in even with groups of around size 5. Further, research in organizational psychology [Axt18] argues that groups of size larger than eight causes conversation quality to erode, people to become less candid, comments to become superficial, and tough topics avoided. The superiority of small groups has also been established via game-theoretic modeling of deliberation [Mei07]. All this means a large group interaction, even with the best intentions, is unlikely to find something as optimal as their 1-median outcome, or allow participants to be "public spirited" [FPW23] and compute the true social cost of each alternative. What is more likely is that the large group splits into multiple groups of constant size whose deliberation outcomes are aggregated by a moderator.

1.2 Deliberation Model with Small Groups

Although Example 1 may seem to imply a negative result for Question 1 with groups of constant size, our main contribution is the following set of positive results.

> There are natural models for multiple small group deliberations with constant-sized groups and natural social choice rules for aggregating their ordinal outcomes, which break the lower bounds of metric distortion (without deliberation) even with groups of size $k = 3$ or 4, and where the distortion is $1 + O(1/\text{poly}(k))$ for larger group sizes k.

For these results, we go beyond each deliberating group producing the 1-median alternative. We do so by restricting the set of alternatives that a group can deliberate on, so that different small groups deliberate on different, focused sets of alternatives. In particular, we let each group deliberate on only two alternatives and output the better of the two. We then choose multiple groups of size k at random from the population and let them deliberate between different pairs of alternatives, with the result of the deliberation being either the first or the second alternative. The outcome of these deliberations produces a tournament graph over the alternatives, and we use the classic Copeland rule [Szp10] to aggregate this tournament into a single winning alternative. The entire model is presented in Section 2.

For analyzing this framework, we assume that the outcome of any individual deliberation is a (possibly probabilistic) function of the strengths of the preference of voters within the group (as measured by their metric distances) for the two alternatives they are deliberating over. We consider two models: In the *averaging model*, the group deliberates between the two alternatives and outputs that alternative whose 1-median cost is smaller.[1] In the *random choice model*, each group member has a normalized bias (or strength

[1] In this model, instead of any group of size k deliberating only over a pair of alternatives, we can equivalently let the group deliberate over all the alternatives and output a ranking based on their total distance to each alternative.

of preference) for one of the two alternatives proportional to the difference in their distance to the two alternatives. The probability with which an alternative is produced is proportional to the total normalized bias of voters in the group who prefer that alternative to the other. Note that in both cases, the output of a group is an ordinal preference.

1.3 Our Results

We show analytic results for the distortion of both the averaging and random choice models as a function of the group size k, showing that a very small group size suffices to break the lower bound of metric distortion without deliberation. In particular, we show the following results, which are also summarized in Tables 1 and 2.

Group size	Upper Bound	Lower Bound
$k = 2$	$3 + \sqrt{2} \approx 4.414$ (Theorem 3.3)	$1 + \sqrt{2} \approx 2.414$ (Lemma 3.5)
$k = 3$	2.81 (Theorem 3.7)	$\frac{5}{3} \approx 1.667$ (Theorem 3.2)
General k	$1 + O\left(\frac{1}{k}\right)$ (Theorem 3.1)	$1 + \Omega\left(\frac{1}{k}\right)$ (Theorem 3.2)

Table 1: The upper and lower bounds on distortion for the Averaging model of deliberation. The upper bounds hold for the Copeland rule, while the lower bounds hold for any deterministic social choice rule that aggregates the ordinal rankings output by all groups of size k. The upper bound of $3 + \sqrt{2} \approx 4.414$ is tight for the Copeland rule when $k = 2$.

Group size	Upper Bound (Theorem 4.1)	Deterministic LB	Randomized LB
$k = 2$	3.34	1.82	1.41
$k = 3$	2.31	1.51	1.25
$k = 4$	1.90	1.37	1.18

Table 2: The upper and lower bounds on distortion for the Random Choice model of deliberation. The upper bounds hold for the Copeland rule. The final two columns respectively show the lower bounds for any deterministic and randomized social choice rule that aggregates the ordinal rankings output by all groups of size k. These follow from Theorem 2.3. For general k, the upper bound is $1 + O\left(\sqrt{\frac{\log k}{k}}\right)$ (Theorem 4.2), and approaches 1 as $k \to \infty$.

- For the averaging model with group size $k = 3$, the distortion bound is 2.81 (Theorem 3.7). This bound is smaller than the deterministic metric distortion lower bound of 3. (As we discuss in Section 2, the fair comparison for the averaging model is deterministic social choice rules.) For the random choice model, the distortion is 1.90 for $k = 4$ (Theorem 4.1), which is lower than the lower bound of randomized metric distortion of 2.11. We also show that for $k = 2$, the distortion is exactly $3 + \sqrt{2} \leq 4.42$ for the averaging model (Theorem 3.3) and at most 3.34 in the random choice model. The proofs use interesting non-convex programming relaxations for which we computationally bound the global optimum.[2]

[2] All Python code is available at http://bit.ly/3WHW8H4

- Next, we consider the asymptotic behavior of distortion in the size of the group, showing that the distortion approaches 1 rapidly as the size of the group increases. For the averaging model, we show that groups of size $k = \Theta(1/\varepsilon)$ are necessary and sufficient to achieve distortion $1 + \varepsilon$, where $\varepsilon > 0$ is a constant (Theorems 3.1 and 3.2). This result is an interesting application of the Berry-Esseen inequality [Fel71]. For the random choice model, we show an upper bound of $k = \tilde{O}(1/\varepsilon^2)$ (Theorem 4.2). These bounds on group size are *independent* of the number of alternatives, a result that cannot be obtained had each group only output its favorite alternative among all m alternatives, for example, as in [CMP24].

- Our analytic results assume a continuum of users from which infinitely many groups of size k can be sampled. We also consider "sample complexity", which is the number of randomly sampled groups needed to achieve an additive ε to the analytically computed distortion. We show that the sample complexity is $O\left(\frac{\log m}{\varepsilon^2}\right)$ for the averaging model (Theorem 3.9), and $O\left(\frac{m \log m}{\varepsilon^2}\right)$ for the random choice model (Theorem 4.3).

- In Section 4.4, we present generalizations of the random choice model that are motivated by how the change in voter opinion depends on bias. We show that our analysis smoothly extends to these more general models, where the bounds become worse (asymptotically converging to numbers larger than one) and closer to the distortion of random dictatorship as the opinion change becomes less dependent on the bias of the voter.

From a practical perspective, our protocols and associated distortion bounds show that multiple tiny groups of people deliberating about specific alternatives suffice to improve on straightforward voting over all alternatives. This result is robust to natural models of deliberation. Since the goal of the paper is to make a conceptual point that very small group sizes suffice, we have focused on the Copeland rule. However, some of our analysis (such as Theorem 3.3) can be extended to weighted tournament rules [MW19, Kem20], with comparable guarantees.

1.4 Technical Highlight: Small Deviation Bounds and Non-convex Programs

It is known that the distortion of the Copeland rule can be characterized by its behavior on at most 3 alternatives [ABE$^+$18]; most of our analysis uses its behavior on two alternatives. As one of our contributions, we distill the type of analysis in [ABE$^+$18] as solving a mathematical program about distributions. At a high level, such a mathematical programming approach is akin to the approach in [CR22], though the details are very different. For the averaging model, the problem of bounding the distortion reduces to a very basic question in probability theory (see Eq. (2) in Section 3):

Question 2. Given $k \geq 1$, find the distribution D_k supported on $[-1, 1]$ with maximum $\mathbb{E}[D_k]$, subject to the constraint that for $Y_1, Y_2, \ldots, Y_k \overset{\text{i.i.d.}}{\sim} D_k$, we have Median $\left(\frac{\sum_{i=1}^k Y_i}{k}\right) \leq 0$.

Here, k is the group size. We call the optimum expectation computed above θ_k. This captures the worst-case gap between the mean and median of the average of bounded *i.i.d.* random variables. We show that the distortion is both upper and lower bounded by simple increasing functions of θ_k, so that tightly bounding θ_k yields good bounds on distortion. As $k \to \infty$, by the Central Limit Theorem, the average of $i.i.d.$ and bounded random variables converges to a Normal distribution, so that $\theta_k \to 0$. Our goal is to bound θ_k for *small* k, and bound the asymptotic rate at which θ_k decreases with k. Our main technical result in Section 3 is summarized in the theorem below and captures the combination of the bounds in Theorems 3.1, 3.2 and 3.7 and Lemma 3.5. This theorem may be of independent interest in probability theory.

Theorem 1.1 (Proved in Section 3.). *We have the following bounds*[3] *on* θ_k*:*

$$\theta_2 = \sqrt{2} - 1; \qquad 0.25 \le \theta_3 \le 0.2522; \qquad \textit{and} \qquad \theta_k = \Theta\left(\frac{1}{k}\right).$$

Though the above problem is easy to solve for groups of size $k = 1$, even with groups of size $k = 2$, computing θ_2 involves a non-convex convolution constraint over a distribution with infinite support, whose expectation we need to globally optimize. Our main contribution is techniques to reduce the size of these programs via near-tight non-convex relaxations, so that a state of the art non-linear optimizer (BARON [Sah96, KS18]) can globally optimize it. With groups of size $k = 2$, we use pipage rounding [AS04] combined with a relaxation to reduce the support size of the optimal distribution. For $k = 3$, we use a technique in [Fei04] to replace the identical random variables $\{Y_i\}$ with nonidentical ones, each with small support. For $k \ge 4$, the problem becomes challenging. Indeed, the complexity of our relaxation grows significantly even in going from $k = 2$ to $k = 3$ and it is an open question whether there are tractable approaches even for $k = 4$ that show reasonably tight bounds.

Conceptually, our mathematical programs have constraints on tail bounds of sum of independent random variables. However, we are interested in the median deviating from the mean, which corresponds to *small deviations* as opposed to large deviations (since the median is at most one standard deviation from the mean [MU17]). There is scant work on small deviation bounds, the exceptions being the elegant works of [Fei04, BP05] (see also [HZZ10]), and the works bounding the location of the median for the sum of special types of distributions [Sie01, KB80, CR86]. It is well-known that moment-based methods such as Bernstein's inequality [BLM13] are slack in this regime, and in our setting, even tighter third moment bounds [BP05] only yield weak results. This necessitates our non-convex programming approach.

To tightly bound the asymptotic dependence of θ_k on k, we use the Berry-Esseen inequality [Fel71], which bounds the deviation in the CDF of the average of *i.i.d.* random variables from a Normal distribution. This is a substantial improvement over the $O(1/\varepsilon^2)$ bounds that are often seen in social choice (e.g. [CMP24]) and cannot be achieved with moment-based methods, since unlike those methods, the Berry-Esseen inequality *becomes tighter* with increasing variance of the underlying random variables.

For the random choice model in Section 4, we show that the constraints in the non-convex program have a certain convexity structure that can be exploited to relax the number of variables to 2 via Jensen's inequality. Though the relaxed program is still non-convex, we can then perform a parametric search over the two variables to find the global optimum. This allows us to compute and reasonably tightly bound the distortion for all small $k \ge 2$ in Fig. 1a.

1.5 Related Work

Our model relates to three strands of recent research and we compare with them below.

Metric Distortion of Voting. Building on the work of [PR06], the work of [ABE+18] initiated the study of metric distortion. Given a social choice rule with ordinal voter preferences that are consistent with an underlying metric over voters and candidates, the distortion is the worst-case (over all metrics) ratio of the total distance to the chosen alternative to that for the 1-median alternative had the metric been known. They showed that the Copeland rule has distortion 5, and no deterministic voting rule has distortion better than 3. The upper bound was subsequently improved via weighted tournament rules to $2 + \sqrt{5}$ by [MW19, Kem20], and later to 3 by [GHS20, KK22]. For randomized voting rules, the work of [AP17] showed a lower bound of 2. This lower bound was subsequently improved to 2.11 by [CR22]. An upper

[3]The work of [ABE+18], as well as a direct analysis implies $\theta_1 = 0.5$, and is achieved by $D_1 = \text{Bernoulli}(1, 1/2)$.

bound of 3 follows from random dictatorship [AP17], and this was improved via novel voting rules to 2.74 in [CWRW24]. We refer the reader to [AFSV21] for a survey.

Our work shows that simple models of deliberation in very small groups ($k = 3, 4$), where the output of the group is ordinal, yet based on their collective cardinal preferences, suffices to allow the Copeland rule to break the metric distortion lower bounds, and the distortion approaches 1 as the group size increases. In the absence of deliberation, "optimal" rules such as Plurality Veto [KK22] have better distortion than Copeland. However, these rules always output the favorite candidate of some voter. If such a rule is used to aggregate the outcome of deliberations, the distortion will be 3 for any constant group size (see Example 1). This showcases a nice property of the Copeland rule, along with its relative ease of analysis.

Voting with Cardinal Information. Our work shows that if groups of voters output ordinal preferences over alternatives via aggregating their cardinal metric information using deliberation, the resulting ordinal information can be aggregated in a way that achieves distortion bounds arbitrarily close to 1. In our models, the outcome of deliberation favors voters with large bias towards one outcome, which is also where the median in the metric space between these voters will likely lie.

The work of [AAZ19] is similar in spirit in that it tags voters' preferences between pairs of alternatives with "strong" and "weak", counting each strong preference by a fixed larger amount than a weak preference. However, in their model, the threshold between strong and weak preference is an arbitrary threshold on the ratio between distances to the two alternatives, and the score bump is also an arbitrary number. Even with an optimal setting of these thresholds, they are unable to break the metric distortion lower bound of 3. This is because the voting rule only gets the original ordinal information had all voters had strong or weak preferences. Our model has two advantages. First, it does not have arbitrary thresholds. Second, in our model, voters within a group use cardinal information optimally via deliberation to find an ordinal ranking between two outcomes (say using 1-median). This is not only more natural, but also squeezes more information about the metric space from the voters, so that the distortion asymptotically approaches 1. We also refer the reader to [AFRV22] for a related distributed model for small groups. .

Sampling and Sortition. Several works [FGMS17, FGMP19, FFM20, CMP24] have considered distortion of voting rules when voters are randomly sampled from the population. The random dictatorship mechanism samples one voter, and achieves distortion 3. The work of [FGMS17] shows that if deliberation between two voters is modeled as bargaining with a disagreement alternative, then there is a protocol that achieves low distortion on special types of metrics called *median spaces*. The work of [FGMP19] extends this model to three sampled voters, and bounds the second moment of distortion (and not just the expectation), which for random dictatorship, is unbounded. See also [GL16] for a different protocol and analysis for three voters. The work of [FFM20] shows that sampling k voters leads to bounded the k^{th} moment of the distortion. However, with the exception of [FGMS17], these works assume voters reason with their ordinal preferences, and none of these works improve on the distortion bound of 3. In contrast, we model deliberation in a general metric space as a natural process of reasoning with cardinal information, which leads better distortion bounds with equally small group sizes.

Motivated by citizen assemblies and sortition [IL18, Fis91], the work of [CMP24] considers a model where a large random sample of voters is chosen to deliberate and find a socially optimal outcome, or 1-median. Their work makes a great case for sortition: for example, most citizen assemblies have the number of participants in the several hundreds [WP08, DBM11], and the bounds of $O((\log m)/\varepsilon^2)$ on group size provided by [CMP24] are quite salient in this setting. However, as discussed before, research in psychology and organizational science shows that such large group sizes lead to sub-optimal deliberation. This is a substantial difficulty in applying the results of [CMP24] to design an actual deliberation process. As Example 1 shows, the distortion remains lower bounded by 3 unless the group size becomes logarithmic in the number of outcomes, so this difficulty is not just a matter of improving their analysis. Our distortion bound of $1+O(1/k)$ becomes more salient in this setting since it allows us to use groups of size $O(1/\varepsilon)$ to get distortion bounds of $1+\varepsilon$. Given the impossibility result mentioned above, the process has to change in some way to obtain our bound; hence, breaking down the deliberation process so that each participant takes part in multiple small deliberations as opposed to one large one is key to our result. Thus, our result is not merely a technical improvement over [CMP24], but is qualitatively different. In fact, many existing online platforms for synchronous deliberation (e.g. [sta, myj, mis]) divide the participants into much smaller groups of 3 to 15, a range where our results are salient. Additionally, our results can be composed with those of [CMP24] to simultaneously provide a bound of $O((\log m)/\varepsilon^2)$ on the total number of participants and of $O(1/\varepsilon)$ on the size of each group in a single deliberation.

Similarly, the work of [FPW23] posits a deliberation model where voters incorporate social utility into their rankings. We note that this work also does not posit an interaction mechanism in a small group. Additionally, it focuses on showing that the distortion under social welfare approaches a constant or linear for natural rules (*i.e.*, the social welfare approach starts to achieve bounds similar to those known for metric distortion), whereas we show improved metric distortion bounds. Thus, our results are both different from and complementary to this work as well as that of [CMP24].

Our main contribution is thus an analytical justification for such a model of interaction with very small groups. We note that the overall number of sampled voters needed for distortion close to 1 in the averaging model remains logarithmic in the number of alternatives, hence being comparable to the bound for a single sortition. Our approach therefore presents an alternate view of how sortition can be implemented via multiple small groups, with smoothly improving distortion bounds in the group size.

2 Model and Preliminaries

We now formally present the model and framework for small group interaction. There is a set C of m candidates (or alternatives or outcomes), and $m!$ possible rankings of these alternatives. We assume a continuum of voters, each of whose preference follows one of the rankings.[4] The preferences of a fraction ρ_q of the voters follow ranking q.

In the metric distortion framework, we assume the rankings with $\rho_q > 0$ and the candidates are embedded in a latent metric space with distance function $d(\cdot,\cdot)$. We assume the metric space has discrete (and finite number of) locations, corresponding to the candidates and the rankings with $\rho_q > 0$. The voters whose preferences follow ranking q are placed at a set of locations S_q, where the voter mass at location $i \in S_q$ is ρ_{iq}, with $\sum_{i\in S_q} \rho_{iq} = \rho_q$. We use "metric space" to mean both the distance function, as well as the voter mass at each location. We denote the combination $(d, \vec{\rho})$ as σ. Let Q denote the set of all locations of voters. Given a location $i \in Q$, the ranking corresponding to i is denoted σ_i, and the voter mass there is denoted ρ_i.

Classic Distortion. In the absence of deliberation, we assume the metric space is "consistent" with the rankings. By this, we mean that given location i with $\rho_i > 0$, and two candidates c_1 and c_2, the ranking $q = \sigma_i$ places c_1 higher than c_2 if $d(i, c_1) < d(i, c_2)$. If $d(i, c_1) = d(i, c_2)$, then either of the two candidates can be ranked higher than the other. Note that a consistent metric always exists by placing all rankings with $\rho_q > 0$ at location a and all candidates at location b.

A social choice rule $\mathcal{S}$ takes the rankings with $\rho_q > 0$ (and the corresponding ρ_q) as input, and outputs a single candidate as the "winner". Note that the social choice rule only sees the rankings and the ρ_q, but not the underlying metric space. We also assume the social choice rule is *anonymous* meaning that its

[4] We make the continuum assumption on voters for analytic convenience, noting that the relevant lower bounds for the finite voter metric distortion problem hold in the continuum voter setting as well. (See Theorem 2.3.)

output does not depend on the identities of the alternatives – if the alternatives are permuted by π, the social choice rule outputs $\pi(c)$ if it was originally outputting c.

To measure the quality of the social choice rule, for any candidate c, let $SC(c, \sigma) = \sum_{i \in Q} \rho_i d(i, c)$ denote the social cost of c under the metric $\sigma = (d, \vec{\rho})$. Let $c^*(\sigma) = \text{argmin}_{c \in C} SC(c, \sigma)$ be the *social optimum* or the median outcome had the metric space σ been known. This outcome need not be unique, in which case an arbitrary such outcome is output. Let $\mathbf{Alg}(\mathcal{S}, \vec{\rho})$ denote the output of the social choice rule $\mathcal{S}$ given the rankings q and their masses ρ_q. Then the distortion of the rule is

$$\text{Distortion } = \max_{\sigma} \frac{SC(\mathbf{Alg}(\mathcal{S}, \vec{\rho}), \sigma)}{SC(c^*(\sigma), \sigma)},$$

The goal is to find a social choice rule with small (preferably constant) distortion.

2.1 Deliberation Models

In our deliberative framework, there is a group size parameter $k \geq 2$. A single deliberation is a function that operates over two alternatives c_1 and c_2, and k randomly chosen voters from the continuum of voters, and outputs one of the alternatives. By "randomly chosen", we mean k rankings are chosen independently with replacement from the set of rankings, with ranking q being chosen at each step with probability ρ_q. Given the continuum assumption on voters, these k rankings (some of which are possibly repeated) correspond to k distinct voters.

In order to define the deliberation function, we first need to define normalized bias. Given a voter at location $i \in Q$, define the *normalized bias* as

$$\mathcal{B}_i(c_1, c_2) = \frac{d(i, c_1) - d(i, c_2)}{d(c_1, c_2)}.$$

The numerator captures the extent of the bias towards one of the two alternatives, with a positive number denoting bias towards c_2. The denominator normalizes this bias by the distance between the two alternatives.

By the triangle inequality $\mathcal{B}(c_1, c_2) \in [-1, 1]$. Note that $\mathcal{B}_i(c_1, c_2) = -1$ implies $d(i, c_2) = d(i, c_1) + d(c_1, c_2)$, so that given $d(i, c_1)$ and $d(c_1, c_2)$, $d(i, c_2)$ is largest possible. This captures the maximum bias of voters towards c_1. Similarly $\mathcal{B}_i(c_1, c_2) = 1$ makes $d(i, c_2)$ as small as possible, and captures strong bias towards c_2. Note that when $\mathcal{B}(c_1, c_2) = 0$, then $d(i, c_1) = d(i, c_2)$, so the voter is indifferent between the two outcomes.

The alternative that is output by the deliberation is a function of these normalized biases for the k voters. We consider the following two models. In both these models, we let S be the multi-set of locations of the voters who participate in the deliberation, so that $|S| = k$.

Averaging Model. (Section 3.) Let $\tau = \sum_{i \in S} \mathcal{B}_i(c_1, c_2)$. If $\tau < 0$, the outcome is c_1, else it is c_2. This corresponds to voters choosing the median, $\text{argmin}_{c=c_1,c_2} \sum_{i \in S} d(i, c)$. This model is equivalent to the model where the group deliberates over all m alternatives, and outputs the ranking consistent with the total distance, so that c_1 is ahead of c_2 in the ranking if $\sum_{i \in S} d(i, c_1) \leq \sum_{i \in S} d(i, c_2)$. As before, in the case of a tie, we assume an arbitrary tie-breaking rule between the alternatives.

Random Choice Model. (Section 4.) In this model, let $S_1 \subseteq S$ denote the set of voters who perfer c_1 to c_2, and let $S_2 \subseteq S$ denote those that prefer c_2 to c_1. Let $A = \sum_{i \in S_1} |\mathcal{B}_i(c_1, c_2)|$ denote the total normalized bias of voters preferring c_1, and let $B = \sum_{i \in S_2} |\mathcal{B}_i(c_1, c_2)|$ be that for c_2.[5] Then the

[5]Since S is a multiset, if a location i is repeated ℓ times, its contribution to the summation is $\ell \cdot |\mathcal{B}_i(c_1, c_2)|$.

output is c_1 with probability $\frac{A}{A+B}$, and c_2 otherwise. We assume the metric is perturbed slightly so that at least one $\mathcal{B}_i(c_1, c_2) \neq 0$; alternatively, if all $\mathcal{B}_i(c_1, c_2) = 0$, we assume c_1 is chosen.

In Section 4, we also generalize the model so that there is a concave, non-decreasing function $g(x)$ with $g(0) = 0$ and $g(1) = 1$. We use $A = \sum_{i \in S_1} g(|\mathcal{B}_i(c_1, c_2)|)$ and $B = \sum_{i \in S_2} g(|\mathcal{B}_i(c_1, c_2)|)$ in randomly choosing the outcome.

The random choice model captures the intuition that more biased voters are likely to drive the discussion in their direction, though that outcome is far from certain. This model can also be interpreted as opinion change as follows: During deliberation, suppose every voter in S_1 independently changes their opinion to c_2 with probability $\frac{B}{A+B}$, and similarly, every voter in S_2 changes their opinion to c_1 with probability $\frac{A}{A+B}$. This process corresponds to the DeGroot model of opinion change [DeG74]. If subsequently, the opinion of a random voter is implemented, this exactly corresponds to the random choice model. A final, somewhat non-deliberative interpretation is that the process chooses a voter proportional to their bias (which might be proportional to how much they speak up), and outputs that voter's preference.

Remark. For $k = 1$, both models reduce to a single voter choosing the candidate among c_1, c_2 that is higher in their ranking. Note also that in both these models, the outcome of deliberation is simply an ordinal preference between the two alternatives. The voters arrive at this alternative by reasoning about the underlying metric space, but this reasoning is hidden from the social choice rule that we describe next.

2.2 Normalized Bias and the Distortion of Copeland's Rule

We now define the social choice rule to aggregate the outcome of deliberations. Note that each deliberation is a probabilistic mapping of a pair of alternatives to a winning alternative. This mapping depends on the underlying metric space as well as the k randomly sampled voters.

Given a deliberation model $\mathcal{A}$ and metric space $\sigma = (d, \vec{\rho})$, let $p_k(W, X)$ denote the probability that the outcome of deliberation among a set S of k randomly chosen participants between alternatives W and X, outputs W instead of X. This probability is over both the randomness in the choice of S and the randomness in the outcome of the deliberation given S (as in the random choice model). To keep our analysis simple, we assume the probability can be exactly estimated for any (W, X); in other words, we assume infinitely many groups of size k can be drawn from the population, and small group deliberations run between W and X. In Theorems 3.9 and 4.3, we will remove this assumption and analyze the number of groups of size k that need to be sampled to approximately achieve the same distortion.

Copeland Rule and Distortion. The Copeland rule dates back to Ramon Llull in the 13^{th} century [Szp10]. To define this rule, we say that $W \succ X$ if $p_k(W, X) \geq 1/2$. For every pair (X, Y), the Copeland rule gives 1 point to X (resp. Y) if $p_k(X, Y) > 1/2$ (resp. $p_k(Y, X) > 1/2$), and half a point to each alternative if $p_k(X, Y) = 1/2$. It then chooses the alternative W with most points. Such an alternative W belongs to the *uncovered set* [Mil77]: For any other alternative X, either $W \succ X$, or there is an alternative Y such that $W \succ Y$ and $Y \succ X$. We assume an arbitrary alternative in the uncovered set is output. Note that the Copeland rule operates over the ordinal outcomes of the deliberations, without knowing the metric space σ.

To define distortion, let $\mathbf{Alg}(\mathcal{A}, \sigma)$ denote the outcome of applying Copeland to the tournament graph on deliberations using model $\mathcal{A}$. As before, let $c^*(\sigma) = \text{argmin}_{c \in C} SC(c, \sigma)$ denote the social optimum. Then the distortion of the deliberation rule $\mathcal{A}$ under the Copeland rule is:

$$\text{Distortion}(\mathcal{A}) = \max_{\sigma} \frac{SC(\mathbf{Alg}(\mathcal{A}, \sigma), \sigma)}{SC(c^*(\sigma), \sigma)}.$$

If the outcome $\mathbf{Alg}(\mathcal{A}, \sigma)$ is probabilistic, we replace the numerator in the above expression by its expectation over the randomness in $\mathcal{A}$.

The Averaging Model is Deterministic. As an aside, suppose there is a finite number n of voters, where n is suitably large. Then, in the Averaging model, the randomness in the choice of set S can be replaced by considering *all* subsets S of k voters, and computing $r_k(W, X)$ as the fraction of these sets where deliberation between alternatives W and X leads to W being chosen. It is easy to check that as $n \to \infty$, the value $r_k(W, X)$ converges to $p_k(W, X)$. Note that the Averaging model itself is a deterministic process given S; further, the Copeland rule applied to the $\{q_k(W, X)\}$ is deterministic. We therefore compare the distortion of the Averaging model with that of *deterministic* social choice rules without deliberation. In the subsequent discussion, we will go back to assuming there is a continuum of voters, and S is a randomly chosen set of k rankings.

Mathematical Program for Distortion. We now show a simple mathematical program whose optimal solution yields an upper bound on distortion of the Copeland rule.

Fix a metric $(d, \vec{\rho})$, and consider two alternatives W and X. Let $\phi_i = \mathcal{B}_i(W, X)$ be the normalized bias of voters at location i, and let D denote the distribution of ϕ, so that $\phi = \phi_i$ with probability ρ_i. Since we fix the metric space, we omit it from the notation $SC(W)$, etc. We have the following lemma, which formulates the analysis of the Copeland rule in [ABE+18] as the solution to a mathematical program.

Lemma 2.1. *Let $\gamma = \mathbb{E}[D]$. Then,*

$$\frac{SC(W)}{SC(X)} \leq \frac{1+\gamma}{1-\gamma}.$$

Proof. Assume $d(W, X) = 1$. We have

$$SC(W) - SC(X) = \sum_i \rho_i \left(d(i, W) - d(i, X)\right) = \sum_i \rho_i \phi_i = \mathbb{E}[D] = \gamma.$$

Next, by triangle inequality, $d(i, X) \geq d(X, W) - d(i, W)$, so that

$$SC(X) = \sum_i \rho_i d(i, X) \geq \frac{1}{2} \sum_i \rho_i \left(d(X, W) - (d(i, W) - d(i, X)\right) = \frac{1}{2}\left(1 - \gamma\right).$$

Manipulating the above two inequalities completes the proof. □

We now find the worst case of γ over all metrics $\sigma = (d, \vec{\rho})$. For this, let $q(W, X)$ denote the solution to the following optimization problem: Find a distribution $D = \{\rho_i\}$ over $\{\mathcal{B}_i(W, X)\}$ which optimizes:

$$\max_D \mathbb{E}[D] \qquad \text{s.t.} \qquad W \succ X, \tag{1}$$

Since $\mathcal{B}_i(W, X) \in [-1, 1]$, the above can be reformulated as optimizing over all distributions D supported on $[-1, 1]$. We next show that upper-bounding this optimum suffices to upper-bound the distortion.

Theorem 2.2. *For deliberation model $\mathcal{A}$, if $q(W, X) \leq \theta$ for all pairs W, X of alternatives, then the distortion of the Copeland rule applied on the outcomes of $\mathcal{A}$ is at most $\left(\frac{1+\theta}{1-\theta}\right)^2$.*

Proof. Fix some metric $(d, \vec{\rho})$. Suppose the Copeland rule finds outcome W, and suppose X is the social optimum. In the case where $W \succ X$, by the previous lemma, the distortion is

$$\frac{SC(W)}{SC(X)} \leq \frac{1+\theta}{1-\theta}.$$

Otherwise, there exists Y such that $W \succ Y$ and $Y \succ X$. Then, the distortion is

$$\frac{SC(W)}{SC(X)} = \frac{SC(W)}{SC(Y)} \cdot \frac{SC(Y)}{SC(X)} \leq \left(\frac{1+\theta}{1-\theta}\right)^2.$$

Since these bounds hold for all metrics $(d, \vec{\rho})$, the proof is complete. □

Given the above theorem, for any deliberation model $\mathcal{A}$, our goal is to upper bound the optimum $q(W, X)$ in Eq. (1) for any pair of alternatives. This will imply an upper bound on distortion. Since this upper bound is independent of the alternatives themselves (as it simply optimizes over all distributions supported on $[-1, 1]$), we only need to show it for an arbitrary pair of outcomes. We will do this in the next sections for the averaging and the random choice deliberation models.

Before proceeding further, we note that solving Eq. (1) also provided a lower bound for distortion.

Theorem 2.3. *For $k \geq 2$, if Eq. (1) has value θ_k, then the distortion of any deterministic social choice rule is at least $\min\left(3, \frac{1+\theta_k}{1-\theta_k}\right)$, and that of any randomized social choice rule is at least $\min\left(2, \frac{1}{1-\theta_k}\right)$.*

Proof. Consider an instance with two alternatives W and X that are distance 1 apart on a line. Note that $W \succ X$ so that $p_k(W, X) = \eta \geq 1/2$.

First consider deterministic rules. Suppose such a rule outputs X. Then we place the η mass of voters preferring W to X at W and the $1 - \eta$ mass of voters preferring X to W at distance $1/2$ from both alternatives. Then $\frac{SC(X)}{SC(W)} = \frac{1+\eta}{1-\eta} \geq 3$. We therefore assume the rule outputs W. In this case, consider the distribution D that yields the optimum solution to Eq. (1). Suppose $\Pr[D = a] = p_a$ for $a \in [-1, 1]$. We place voters of mass p_a at distance $\frac{1+a}{2}$ from W and $\frac{1-a}{2}$ from X. Note that $SC(X) = \frac{1-\mathbb{E}[D]}{2}$, and $SC(W) = \frac{1+\mathbb{E}[D]}{2}$, which implies the distortion is at least $\frac{1+\theta_k}{1-\theta_k}$.

Similarly, for randomized rules, let α denote the probability the rule outputs W. If $\alpha \leq 1/2$, then consider the instance that places η mass of voters preferring W to X at W and the $1 - \eta$ mass of voters preferring X to W at distance $1/2$ from both alternatives. The distortion is $\frac{\alpha \cdot SC(W) + (1-\alpha) \cdot SC(X)}{SC(W)} \geq \frac{1}{1-\eta} \geq 2$. We therefore assume $\alpha \geq 1/2$. Again consider the worst-case distribution D and the instance that places voters of mass p_a at distance $\frac{1+a}{2}$ from W and $\frac{1-a}{2}$ from X. The social optimum is X and it is easy to check that the expected distortion is minimized when $\alpha = 1/2$, and is at least $\frac{1}{2} + \frac{1}{2} \cdot \frac{1+\theta_k}{1-\theta_k} = \frac{1}{1-\theta_k}$. □

3 Distortion in the Averaging Model

We first consider the averaging model. We show in Theorem 3.1 that the group size of $k = \Theta(1/\varepsilon)$ is both necessary and sufficient for distortion $1 + \varepsilon$. In this model, it is analytically difficult to tightly bound distortion for small k; nevertheless, we show that the distortion is tight at $3 + \sqrt{2} \approx 4.414$ for groups of size $k = 2$ (Theorem 3.3), while it is at most 2.81 for groups of size $k = 3$ (Theorem 3.7). This shows that a very small group size ($k = 3$) suffices to beat the deterministic metric distortion lower bound of 3 in this model. (Note that by the discussion in Section 2, the Averaging model can be viewed as a deterministic social choice rule.) We finally show in Theorem 3.9 that the number of sampled groups needed to achieve an additive δ approximation to this bound is only $\tilde{O}(\log m)$, where m is the number of alternatives.

In the averaging model, we choose a set of k random voters to deliberate between two alternatives W and X. Let the multiset of their locations be denoted by S. We compute $\sum_{i \in S} \mathcal{B}_i(W, X)$. If this is negative, the outcome of the deliberation is W, else it is X. This corresponds to outputting $\text{argmin}_{c \in \{W,X\}} \sum_{i \in S} d(i, c)$. This model is equivalent to the model where the group deliberates over all m alternatives, and outputs the ranking consistent with the total distance. This implies c_1 is ahead of c_2 in the ranking if $\sum_{i \in S} d(i, c_1) \leq \sum_{i \in S} d(i, c_2)$, and our analysis extends directly to that case.

We now specialize Eq. (1) for this model. Suppose there are two alternatives W, X between which the deliberation happens. Let $\phi_i = \mathcal{B}_i(W, X)$. The event of W being the winner maps to the condition $\sum_{i \in S} \phi_i \leq 0$. Therefore, we seek a distribution D whose support is $[-1, 1]$ that solves the following problem:

$$\theta_k := \max_D \mathbb{E}[D] \qquad \text{s.t.} \qquad \Pr_{\{a_1, a_2, \ldots, a_k\} \sim D} \left[\sum_{i=1}^{k} a_i \leq 0 \right] \geq 1/2. \tag{2}$$

where the probability is over a set of k independent samples $a_1, \ldots, a_k$ drawn from D.

3.1 Asymptotic Behavior of Distortion in Group Size k

We will now use the Berry-Esseen theorem [Fel71] to show a tight bound of on the asymptotic behavior of θ_k in k. This will yield the following theorem.

Theorem 3.1. *For any $k \geq 2$, $\theta_k = O\left(\frac{1}{k}\right)$. By Theorem 2.2, this means the distortion of the Copeland rule in the averaging model of deliberation using groups of size k is $1 + O\left(\frac{1}{k}\right)$. In particular, for any $\varepsilon > 0$, groups of size $k = O\left(\frac{1}{\varepsilon}\right)$ are sufficient for distortion $1 + \varepsilon$.*

Proof. Let $\theta_k = \mu = \mathbb{E}[D] \geq 0$, and let $D_k = \frac{\sum_{i=1}^{k} a_i}{k}$, so that $\mathbb{E}[D_k] = \mu$. Let $\sigma^2 = \text{Var}[D]$. We have $\text{Var}[D_k] = \sigma^2/k$. The condition $W \succ X$ implies $\Pr[D_k \leq 0] \geq 1/2$, which means the median is at most 0. Noting that the gap between the median and mean of any distribution is at most the standard deviation [MU17], this implies

$$\text{Var}[D_k] = \frac{\sigma^2}{k} \geq (0 - \mu)^2 \qquad \Rightarrow \qquad \sigma \geq \mu\sqrt{k}.$$

Since $\sigma \leq 1$ for a random variable bounded in $[-1, 1]$, this already implies $\theta_k = \mu \leq \frac{1}{\sqrt{k}}$.

We will now improve this bound using the Berry-Esseen inequality [Fel71] applied to the average of k random variables distributed as $D - \mu$. Note that $\mathbb{E}[D - \mu] = 0$, and $\mathbb{E}[(D - \mu)^2] = \text{Var}[D] = \sigma^2$. Further since $\mu, |D| \leq 1$, we have $\mathbb{E}[|D|^3] \leq \mathbb{E}[D^2]$, so that

$$\begin{aligned}
\mathbb{E}[|D - \mu|^3] &\leq \mathbb{E}[(|D| + \mu)^3] \\
&= \mathbb{E}[|D|^3] + 3\mu \cdot \mathbb{E}[D^2] + 3\mu^2 \cdot \mathbb{E}[|D|] + \mu^3 \\
&\leq 4 \cdot \mathbb{E}[D^2] + 4 \cdot \mu^2 \\
&= 4 \cdot \sigma^2 + 8 \cdot \mu^2
\end{aligned}$$

Let $Y = \frac{(D_k - \mu)\sqrt{k}}{\sigma}$, so that $\mathbb{E}[Y] = 0$ and $\text{Var}[Y] = 1$. The condition $W \succ X$ implies

$$\Pr\left[Y \leq \frac{-\mu\sqrt{k}}{\sigma}\right] \geq 1/2.$$

Let $F_Y(z)$ denote $\Pr[Y \leq z]$, and $\Phi(z)$ denote the CDF of the standard Normal distribution. Let $z^* = -\frac{\mu\sqrt{k}}{\sigma}$. By the Berry-Esseen theorem (whose preconditions can be checked to apply here) applied at the point z^*, we have:

$$\frac{1}{2} - \Phi(z^*) \leq F_Y(z^*) - \Phi(z^*) \leq \frac{0.5 \cdot E[|D - \mu|^3]}{\mathbb{E}[(D - \mu)^2] \cdot \sqrt{k}} \leq \frac{2\sigma^2 + 4\mu^2}{\sigma^3\sqrt{k}}.$$

where we have used $\mathbb{E}[|D - \mu|^3] \leq 4\sigma^2 + 8\mu^2$.

Next note that by the above lower bound on σ, we have $z^* \geq -1$, so that $\phi(z^*) \geq \frac{1}{\sqrt{2\pi e}}$, where ϕ is the PDF of the standard Normal distribution. We therefore have

$$\frac{1}{2} - \Phi(z^*) = \Phi(0) - \Phi(z^*) \geq \phi(z^*) \cdot (0 - z^*) \geq \frac{1}{\sqrt{2\pi e}} \cdot \frac{\mu\sqrt{k}}{\sigma}.$$

Putting the above two inequalities together, and using $\frac{\mu^2}{\sigma^2} \leq \frac{1}{k}$, we have

$$\frac{1}{\sqrt{2\pi e}} \cdot \frac{\mu\sqrt{k}}{\sigma} \leq \frac{2\sigma^2 + 4\mu^2}{\sigma^3\sqrt{k}} \qquad \Rightarrow \qquad \mu \leq 2\sqrt{2\pi e} \cdot \left(\frac{1}{k} + \frac{2}{k^2}\right) \leq \frac{8.27}{k} \cdot \left(1 + \frac{2}{k}\right).$$

Since $\theta_k = \mu$, plugging this bound into Theorem 2.2, the proof of the upper bound is complete. ☐

We now show a matching lower bound, showing our analysis is tight.

Theorem 3.2. *For any $k \geq 2$, $\theta_k = \Omega\left(\frac{1}{k}\right)$. This means that any anonymous social choice rule that aggregates the ordinal rankings of all groups of size k over all alternatives has distortion $1 + \Omega\left(\frac{1}{k}\right)$, so that achieving distortion $1 + \varepsilon$ needs groups of size $\Omega(1/\varepsilon)$.*

Proof. We show the lower bound on θ_k. The bound on distortion follows from Theorem 2.3.

Case 1. Odd k. We place a mass of $1/2$ voters at X, and $1/2$ mass of voters between X and W, at a distance $\frac{2}{k+1}$ from W. The distribution D is therefore 1 with probability $1/2$ and $-1+\frac{2}{k+1}$ with probability $1/2$, so that $\mathbb{E}[D] = \frac{1}{k+1}$. To see that $W \succ X$, if we sample k times from this distribution, the sum of the values is at most 0 as long as the majority of samples are closer to W, which happens with probability exactly $1/2$.

Case 2. Even $k \geq 4$. We place a voter mass of $p = \frac{1}{2} + \frac{1}{3k}$ at X (location 1 in distribution D) and $1 - p = \frac{1}{2} - \frac{1}{3k}$ at W (location -1 in D). This implies $\mathbb{E}[D] = \frac{2}{3k}$. To see that $W \succ X$, let M denote the median of $\texttt{Binomial}(k,p)$, whose mean is $\mu = kp$. The work of [KB80] upper bounds the gap between the mean and median for a Binomial and implies that

$$|M - kp| = \left|M - \frac{k}{2} - \frac{1}{3}\right| \leq \max(p, 1-p) \leq \frac{1}{2} + \frac{1}{3k} \leq \frac{7}{12} \qquad \Rightarrow \qquad \left|M - \frac{k}{2}\right| < 1.$$

Since k is even, this implies $M = \frac{k}{2}$. Consider a sample of k voters. If at least $k/2$ voters are located at W, then W is the winner. But the probability of this event is at least the probability that $\texttt{Binomial}(k,p) \leq M$, which is at least $1/2$. Therefore, $W \succ X$, completing the proof of the lower bound. ☐

In Theorem 3.1, we note that moment based methods like Bernstein's inequality [BLM13] would only show an upper bound of $\theta_k = O\left(\frac{1}{\sqrt{k}}\right)$ – these methods require the variance to be small, while the Berry-Esseen inequality shows the distribution of the average is closer to Normal if the variance is high while the third moment is bounded. Though the latter yields a much stronger bound, it is still far from tight for small k. For instance, it implies that we need a very large constant k to achieve distortion below 3, while we show below that the correct bound is $k = 3$. We next show vastly improved distortion bounds for the canonical cases of $k = 2, 3$. For $k = 3$, we obtain 2.81, which beats the deterministic metric distortion lower bound.

3.2 Distortion Bound for $k = 2$

We now consider the case with groups of size $k = 2$. We note that Theorem 2.3 combined with Lemma 3.5 below shows a lower bound of $1 + \sqrt{2}$ on the distortion of any deterministic social choice rule. We now show that Copeland has distortion $3 + \sqrt{2}$, and this bound is tight. Though the $k = 2$ case sounds simple, even here, we need to solve an interesting non-convex program to global optimality. Note that our bound significantly improves the tight distortion bound of 5 for Copeland when $k = 1$ (no deliberation).

Theorem 3.3. *For the Copeland rule applied to the averaging model with group size $k = 2$, the distortion is at most $3 + \sqrt{2} \leq 4.42$, and this bound is tight for the Copeland rule.*

The rest of this subsection is devoted to proving the above theorem.

Computing θ_2. We first compute θ_2 explicitly. We show the following lemma using the idea of pipage rounding [AS04].

Lemma 3.4. *Consider Eq. (2) with $k = 2$. The optimal D has support $\{-1, 0, 1\}$.*

Proof. To an arbitrarily good approximation, the support of D can be discretized as $\{0, \pm\varepsilon, \pm 2\varepsilon, \ldots, \pm 1\}$. Consider two values $\{-a, a\}$ in the support, where $1 > a > 0$; assume at least one of these values has non-zero mass. Suppose $\Pr[D = a] = p$ and $\Pr[D = -a] = q$, where $\max(p, q) > 0$. Note that the contribution to the objective is $(p - q) \cdot a$. If $p > q$, move the probability mass at a to $a + r \cdot \varepsilon$ and the mass at $-a$ to $-a - r \cdot \varepsilon$; else move the probability mass at a to $a - r \cdot \varepsilon$ and the mass at $-a$ to $-a + r \cdot \varepsilon$. Here, r is the smallest integer so that one of the four values $\pm a \pm r \cdot \varepsilon$ has non-zero mass. It is easy to check that both operations preserve the LHS of the constraint in Eq. (2), since any sum remains least zero iff it was originally least zero. Further, the objective only increases in this process, since the contribution of the moving mass increases, and the remaining mass remains the same as before. We perform that move, thereby eliminating $\{a, -a\}$ from the support. Iterating, all the mass moves to $\{-1, 0, 1\}$. Now taking the limit as $\varepsilon \to 0$, the proof is complete. □

We can now write Eq. (2) as a non-convex optimization problem. Suppose $\Pr[D = -1] = p$, and $\Pr[D = 0] = q$. Then the program becomes:

$$\max_{p,q \geq 0, p+q \leq 1} 1 - 2p - q \qquad \text{s.t.} \qquad (p+q)^2 + 2p(1-p-q) \geq 1/2. \tag{3}$$

We now use the global optimization tool BARON ((Branch-And-Reduce Optimization Navigator) [Sah96, KS18]) [Sah96][6]. BARON uses a combination of convex relaxation and integer programming techniques to find increasingly tighter lower and upper bounds on the global optimum. We find that $q = 0$ and $p = 1 - 1/\sqrt{2}$. Therefore,

Lemma 3.5. $\theta_2 = \sqrt{2} - 1$, *and is achieved when* $\Pr[D = 1] = \frac{1}{\sqrt{2}}$ *and* $\Pr[D = -1] = 1 - \frac{1}{\sqrt{2}}$.

The above lemma implies that when $W \succ X$, we have $\frac{SC(W)}{SC(X)} \leq \frac{1+\theta_2}{1-\theta_2} \leq \sqrt{2} + 1 \approx 2.414$. It also implies a lower bound of $\sqrt{2} + 1$ on the distortion of any deterministic social choice rule by Theorem 2.3.

[6] BARON can be called from Python using the AMPL API (`amplpy`). We have used the demo (free) version available on Google Colab, that supports 10 constraints and variables and up to 50 nonlinear operations. This suffices for all our programs.

Distortion of Copeland. We next bound $\frac{SC(W)}{SC(X)}$ when there exists a Y such that $W \succ Y \succ X$. Towards this end, let $a_i = d(i,W) - d(i,Y)$ and $b_i = d(i,Y) - d(i,X)$. Let $d(Y,X) = 1$, and $d(W,X) = B \geq 0$, so that $d(W,Y) \leq B+1$. By triangle inequality, we have $|a_i| \leq B+1$, and $|b_i| \leq 1$ for all locations i.

Let D_1 be the distribution on $\{a_i\}$ induced by D, and D_2 be the corresponding distribution on $\{b_i\}$. The conditions $W \succ Y$ and $Y \succ X$ yield:

$$\Pr_{\{a_1,a_2\}\sim D_1}[a_1 + a_2 \leq 0] \geq 1/2, \text{ and} \tag{4}$$

$$\Pr_{\{b_1,b_2\}\sim D_2}[b_1 + b_2 \leq 0] \geq 1/2, \tag{5}$$

where the two draws from D_1 (resp. D_2) are *i.i.d.* Note that $a_i + b_i = d(i,W) - d(i,X)$ so that

$$SC(W) - SC(X) = \mathbb{E}[a_i + b_i]. \tag{6}$$

We will bound $SC(X)$ as follows. Note that

$$2d(i,X) = d(i,X) + d(i,X) \geq (d(W,X) - d(i,W)) + d(i,X) = d(W,X) - (a_i + b_i).$$

Similarly, we also get

$$2d(i,X) \geq d(Y,X) - b_i.$$

Therefore,

$$SC(X) \geq \frac{1}{2} \cdot \mathbb{E}\left[\max(B - a_i - b_i, 1 - b_i)\right]. \tag{7}$$

To show that the distortion is at most $1+\beta$ it therefore suffices to show

$$SC(W) - SC(X) \leq \beta \cdot SC(X).$$

Plugging in Eqs. (6) and (7), and simplifying, it suffices to show that the following objective is at most 0.

$$\eta_2 := \text{Maximize}\left(\mathbb{E}\left[\left(\frac{2}{\beta}+1\right)b_i\right] + \mathbb{E}\left[\frac{2}{\beta}\cdot a_i + \min\left(a_i - B + 1, 0\right) - 1\right]\right) \tag{8}$$

where the maximization is over B, and distributions D_1, D_2, where D_1 is a distribution over $\{a_i\}$ supported on $[-B-1, B+1]$, and D_2 is a distribution over $\{b_i\}$ supported on $[-1,1]$. The maximization is subject to constraints Eqs. (4) and (5). We will now show that the above maximum is at most 0.

Note that the optimum for $\{b_i\}$ subject to Eq. (5) and $|b_i| \leq 1$ and that for $\{a_i\}$ subject to Eq. (4) and $|a_i| \leq B+1$ can be separately computed. For $\{b_i\}$, the optimum is simply $(1 + 2/\beta) \cdot \theta_2$. For $\{a_i\}$, we have the following lemma.

Lemma 3.6. *The optimum distribution D_1 for Eq. (8) is supported on $\{-B-1, B+1, B-1, 1-B, 0\}$.*

Proof. We use the same procedure as in the proof of Lemma 3.4. For any $a > 0$, consider the objective in Eq. (8) for the points $\{a, -a\}$. Let $p = \Pr[D_1 = a]$ and $q = \Pr[D_1 = -a]$, where $\max(p,q) > 0$. The objective is a linear function of a except at the point $a = B-1$, where $-a = 1-B$. To see this, note that the contribution to the objective is both linear in a and linear in $-a$. As an example, if $a > B-1$, then the contribution to the objective is

$$p \cdot \left(\frac{2}{\beta}\cdot a + 0\right) + q \cdot \left(\frac{2}{\beta}\cdot(-a) + (-a - B + 1)\right),$$

which is linear in a. Therefore, we can perform the same operation as in Lemma 3.4 and move the probability mass to the neighboring points, till either it hits one of $B-1, 1-B$, or the extreme points $\{-B-1, B+1\}$ or 0. Therefore, the optimal D_1 is supported on these five points. □

We can now write Eq. (8) as a non-linear program with six variables, one for B, and the rest capturing the probabilities of D_1 taking one of the five support values. Note that to encode Eq. (4), we need to consider two cases, depending on whether $B \geq 1$ or otherwise. We elaborate below.

Non-convex Program: Case 1. We first consider the case where $0 \leq B \leq 1$. Note that B itself is a variable. Let

$p = \Pr[D_1 = B+1], q = \Pr[D_1 = 1-B], r = \Pr[D_1 = 0], s = \Pr[D_1 = B-1], t = \Pr[D_1 = -B-1].$

These values of D_1 are in decreasing order. Then

$$p + q + r + s + t = 1, \qquad \text{and} \qquad p, q, r, s, t \geq 0. \tag{9}$$

The probability constraint Eq. (4) implies

$$(p+q)^2 + 2p \cdot (r+s) + 2q \cdot r \leq 1/2. \tag{10}$$

Let $A = \left(1 + \frac{2}{\beta}\right) \cdot \theta_2$. Then subject to the above constraints, the objective is:

$$A + p\left(\frac{2}{\beta}(B+1) - 1\right) + q\left(\frac{2}{\beta}(1-B) - 1\right) - r - s\left(\frac{2}{\beta}(1-B) + 1\right) - t\left(\frac{2}{\beta}(B+1) + 2B + 1\right).$$

Non-convex Program: Case 2. We next consider the case where $B \geq 1$, where B is a variable. Let

$p = \Pr[D_1 = B+1], q = \Pr[D_1 = B-1], r = \Pr[D_1 = 0], s = \Pr[D_1 = 1-B], t = \Pr[D_1 = -B-1].$

These values of D_1 are in decreasing order. It is easy to check that constraints Eqs. (9) and (10) remain the same. Let $A = \left(1 + \frac{2}{\beta}\right) \cdot \theta_2$. Then, subject to the above constraints, the objective is now:

$$A + p\left(\frac{2}{\beta}(B+1) - 1\right) + q\left(\frac{2}{\beta}(B-1) - 1\right) - rB - s\left(\frac{2}{\beta}(B-1) + 2B - 1\right) - t\left(\frac{2}{\beta}(B+1) + 2B + 1\right).$$

Note that Eq. (10) implies $p + q \leq \frac{1}{\sqrt{2}}$. This means $r + s + t \geq 1 - \frac{1}{\sqrt{2}}$. Since we assume $\beta \geq 2 + \sqrt{2}$, it is easy to check that the objective is strictly negative if $B \geq 100$. Therefore, add $0 \leq B \leq 100$ as a constraint. We find the global optimum via BARON [Sah96, KS18] for both cases. For $\theta_2 = \sqrt{2} + 1$, and $\beta = 2 + \sqrt{2} + \delta$ for small $\delta > 0$, the global optimum for both cases is strictly negative. This shows that $\beta = 2 + \sqrt{2}$, and the distortion is at most $1 + \beta = 3 + \sqrt{2}$, which shows the the upper bound in Theorem 3.3.

Matching Lower Bound. The above program also unearths the worst case example that shows the bound is tight. Place W, X, Y on a line in that order with a distance of 1 between W, X and 1 between X, Y. Place a mass $1 - \alpha$ of voters very close to X and α very close to Y, where $\alpha = 1 - 1/\sqrt{2}$. The distances can be appropriately set so that the following happens: Whenever both voters in a sampled pair are located close to X (which happens with probability $(1-\alpha)^2 = 1/2$), they will prefer W over Y, so that $p_k(W, Y) \geq 1/2$. Next, when one voter is close to X and the other is close to Y, they prefer Y to X. This means $p_k(Y, X) = 1 - (1-\alpha)^2 = 1/2$. This means W is in the uncovered set, and is the Copeland winner. The social optimum is X, and the distortion is $\frac{SC(W)}{SC(X)} = 1 + 1/\alpha = 3 + \sqrt{2}$. This completes the proof of Theorem 3.3.

3.3 Distortion Bound for $k = 3$

We show that groups of size 3 suffice for the distortion to go below the deterministic metric distortion lower bound of 3. (As noted before, the Averaging rule for a finite but large number of voters can be viewed as a deterministic social choice rule.) Note that for $k = 3$, the proof of Theorem 3.2 implies $\theta_3 \geq 0.25$, which also implies a lower bound of $5/3 \approx 1.667$ on the distortion of any deterministic social choice rule via Theorem 2.3. We show that the bound on θ_3 is nearly tight.

Theorem 3.7. *For the optimization problem Eq. (2) with group size $k = 3$, we have $0.25 \leq \theta_3 \leq 0.2522$. By Theorem 2.2, this implies the distortion of the Copeland rule with $k = 3$ is at most 2.81.*

The rest of this subsection is devoted to proving $\theta_3 \leq 0.2522$. We write a compact non-linear programming relaxation for the optimum of Eq. (2). Note that the optimal distribution has expectation θ_3.

The key to our relaxation is the following lemma, which is a corollary of Lemma 1 in [Fei04]. We present a proof for completeness.

Lemma 3.8 ([Fei04])**.** *There exist independent distributions D_1, D_2, D_3 each supported on two values in $[-1, 1]$ (these values could be different for each D_i), such that $\mathbb{E}[D_1] = \mathbb{E}[D_2] = \mathbb{E}[D_3] = \theta_3$, and*

$$\Pr_{a_1 \sim D_1, a_2 \sim D_2, a_3 \sim D_3} [a_1 + a_2 + a_3 \leq 0] \geq 1/2. \tag{11}$$

Proof. We start with the distribution D^* that optimizes Eq. (2). Let Y_1, Y_2, Y_3 denote $i.i.d.$ random variables with distribution D^* so that $a_1 \sim Y_1$, $a_2 \sim Y_2$, and $a_3 \sim Y_3$. First consider Y_1, and assume w.l.o.g. that it is finely discretized as $\{v_1, v_2, \ldots, v_n\}$, and let $p_j = \Pr[Y_1 = v_j]$. Let $q_j = \Pr[Y_2 + Y_3 + v_j \leq 0]$. Then

$$\sum_{j=1}^{n} p_j = 1 \qquad \text{and} \qquad \mathbb{E}[D^*] = \sum_{j=1}^{n} p_j v_j \qquad \text{and} \qquad \Pr[Y_1 + Y_2 + Y_3 \leq 0] = \sum_{j=1}^{n} p_j q_j \geq 1/2.$$

Now treat the p_j as non-negative variables, and maximize the LHS of the third constraint subject to the first two constraints. The optimum has at most two non-zero variables, and satisfies the third constraint (since the LHS cannot decrease). This yields a new distribution D_1 with $\mathbb{E}[D_1] = \theta_3$, and with support two. Repeat this process for Y_2 and then Y_3, completing the proof. □

We will find the optimal (D_1, D_2, D_3) satisfying the above lemma, and this will upper bound θ_3. For $i = 1, 2, 3$, let $D_i = a_i$ with probability $1 - p_i$ and b_i with probability p_i, where for all $i = 1, 2, 3$. Therefore, we have the constraints:

$$b_i \leq a_i; \qquad a_i, b_i \in [-1, 1], \qquad c_i = a_i - b_i, \qquad p_i \in [0, 1] \qquad \forall i = 1, 2, 3. \tag{12}$$

The $\{a_i, b_i, c_i, p_i\}_{i=1}^3$ and θ_3 will be the variables in our program. We also have the constraints:

$$\theta_3 = b_i p_i + a_i(1 - p_i) \qquad \forall i = 1, 2, 3. \tag{13}$$

Our goal is to maximize θ_3. W.l.o.g., enforce the constraint that

$$c_3 \geq c_2 \geq c_1 \geq 0. \tag{14}$$

Write $D_i = a_i - Z_i$ where $Z_i = c_i$ with probability p_i and 0 otherwise. Then the constraint in Eq. (11) translates to:

$$\Pr_{z_1 \sim Z_1, z_2 \sim Z_2, z_3 \sim Z_3} \left[\sum_{i=1}^{3} z_i \geq \sum_{i=1}^{3} a_i \right] \geq 1/2. \tag{15}$$

In sorted order, $\sum_{i=1}^3 z_i$ can take values

$$c_3 + c_2 + c_1 \geq c_3 + c_2 \geq c_3 + c_1 \geq \{c_2 + c_1, c_3\} \geq c_2 \geq c_1 \geq 0.$$

We enforce the constraint that the quantity $\sum_{i=1}^3 a_i$ lies between two of the values; this yields eight possible programs. In each of these programs, we have the constraints Eqs. (12) to (14) and the objective is to maximize θ_3. The cases differ in how Eq. (15) is encoded. These cases are shown in Table 3, where the

Case	Constraint on range of $\sum_{i=1}^{3} a_i$	Constraint Eq. (15)
1	$c_3 + c_2 + c_1 \geq a_3 + a_2 + a_1 \geq c_3 + c_2$	$p_1 p_2 p_3 \geq 1/2$
2	$c_3 + c_2 \geq a_3 + a_2 + a_1 \geq c_3 + c_1$	$p_3 p_2 \geq 1/2$
3	$c_3 + c_1 \geq a_3 + a_2 + a_1 \geq \max\{c_3, c_2 + c_1\}$	$p_3 p_2 + p_3 p_1 (1 - p_2) \geq 1/2.$
4	$c_3 \geq a_3 + a_2 + a_1 \geq c_2 + c_1$	$p_3 \geq 1/2.$
5	$c_2 + c_1 \geq a_3 + a_2 + a_1 \geq c_3$	$p_3 p_2 + p_3(1 - p_2) p_1 + (1 - p_3) p_2 p_1 \geq 1/2$
6	$\min\{c_2 + c_1, c_3\} \geq a_3 + a_2 + a_1 \geq c_2$	$1 - (1 - p_3)(1 - p_1 p_2) \geq 1/2$
7	$c_2 \geq a_3 + a_2 + a_1 \geq c_1$	$1 - (1 - p_3)(1 - p_2) \geq 1/2$
8	$c_1 \geq a_3 + a_2 + a_1$	$1 - (1 - p_3)(1 - p_2)(1 - p_1) \geq 1/2$

Table 3: The possible ranges for $\sum_{i=1}^{3} a_i$ and the encoded constraints for each case.

constraint in the third column encodes Eq. (15) when $\sum_{i=1}^{3} a_i$ lies in the range encoded by the constraint in the second column.

As an example, for the first row, we have $\sum_{i=1}^{3} a_i \in [c_3 + c_2, c_3 + c_2 + c_1]$. But this means for the event in Eq. (15) to be satisfied, we must have $z_3 = c_3, z_2 = c_2, z_1 = c_1$, which happens with probability $p_3 p_2 p_1$. The constraint Eq. (15) therefore implies $p_1 p_2 p_3 \geq 1/2$. We encode the constraints in the second and third columns of the row corresponding to Case (1) into the program for Case (1), and solve it along with constraints Eqs. (12) to (14), and the objective of maximizing θ_3.

For each of the eight cases, we globally maximize θ_3 using BARON [Sah96, KS18]. We observe that in all cases except Case (3), the global optimum is found to be at most 0.25. In Case (3), the solver can only bound it in the range $[0.25, 0.2522]$. Nevertheless, $\theta_3 \leq 0.2522$ in all cases. Plugging this into Theorem 2.2 completes the proof of Theorem 3.7.

We remark that BARON is a state of the art non-linear program solver, and it struggles to find the global optimum even for $k = 3$ (Case 3 above). Given this, we believe an entirely different relaxation will be needed to solve the $k = 4$ case.

3.4 Sample Complexity

So far, our bounds have assumed $p_k(W, X)$ — the probability that a randomly sampled group of size k leads to W as the outcome of deliberation — can be estimated exactly. Via a standard argument, this can be converted to a sampling bound on the number of sampled groups needed to estimate $p_k(W, X)$ for all pairs of alternatives.

For each sampled group of size k, we ask the voters to rank all alternatives. Assuming that c_1 is ranked higher than c_2 if $\sum_{i \in S} d(i, c_1) \leq \sum_{i \in S} d(i, c_2)$, it is easy to check that for each pair (W, X), the outcome of deliberation just among this pair of alternatives will be consistent with the ranking. By a standard application of Chernoff bounds [BLM13], if we sample $O\left(\frac{\log(m/\delta)}{\varepsilon^2}\right)$ groups of size k, then we can approximate each $p_k(W, X)$ to within an additive ε with probability $1 - \delta$. Suppose we output the Copeland winner of the tournament graph on the samples, then with high probability, for the winner W, and for any other alternative X, we either have $p_k(W, X) \geq 1/2 - \varepsilon$, or there exists Y such that $p_k(W, Y) \geq 1/2 - \varepsilon$ and $p_k(Y, X) \geq 1/2 - \varepsilon$. Since θ_k in Eq. (2) is smooth in the RHS of the constraint, this means θ_k is within $O(\varepsilon)$ of the bounds computed above. Plugging this into Theorem 2.2 yields the following theorem.

Theorem 3.9. *Let d_k denote the distortion of the Copeland rule with groups of size k if we could compute $p_k(W, X)$ exactly. Then, for any $\varepsilon > 0$ and $\delta \in (0, 1)$, we have that $O\left(\frac{1}{\varepsilon^2} \cdot \log \frac{m}{\delta}\right)$ randomly sampled groups of size k suffice to achieve distortion $d_k + \varepsilon$ with probability $1 - \delta$.*

As shown in Example 1, if the group size k is a constant and we want to beat the distortion bound of 3, then this crucially requires the deliberating group to rank alternatives *beyond* their favorite (or median) alternative (among all m alternatives). This makes our sampling bound above and in Theorem 4.3 with constant size groups different from core-set type sampling bounds for the 1-median problem, for instance, the bounds in [CMP24].

4 Distortion in the Random Choice Model

We next compute the distortion in the Random Choice model for various values of group size k. In contrast to the averaging model, this model is more analytically tractable, and we can numerically compute a good upper bound on distortion for all small k. In particular, our main result in Theorem 4.1 is that groups of size $k \le 4$ suffice to break the randomized metric distortion lower bound of 2.11. In Theorem 4.2, we also show that the group size is $k = \tilde{O}(1/\varepsilon^2)$ for Copeland to achieve distortion $1+\varepsilon$. We also show in Theorem 4.3 that a sample of $\tilde{O}(m \log m)$ groups suffices for approximating distortion, where m is the number of alternatives. In Section 4.4, we finally present a generalization of the random choice model, and show how our analysis technique naturally extends to showing distortion bounds for it.

4.1 Distortion Bounds for Small k

Suppose the deliberation has k participants, and is between two arbitrary outcomes W and X. For a distribution D, let L be the conditional distribution of $-D$ given $D \le 0$, and let R denote the conditional distribution of D given $D > 0$. Let $\alpha = \Pr[D \le 0]$. The objective in Eq. (1) can be written as

$$\max \ (1-\alpha) \cdot \mathbb{E}[R] - \alpha \cdot \mathbb{E}[L].$$

Let $a_1, a_2 \dots, a_k$ be k i.i.d. samples from L and $b_1, b_2 \dots, b_k$ be k i.i.d. samples from R. Then,

$$p_k(W,X) = \sum_{\ell=1}^{k} \binom{k}{\ell} \alpha^{\ell} (1-\alpha)^{k-\ell} \mathbb{E}\left[\frac{\sum_{r=1}^{\ell} a_r}{\sum_{r=1}^{\ell} a_r + \sum_{q=1}^{k-\ell} b_q}\right],$$

where the expectation is over $a_1, \dots, a_r \overset{\text{i.i.d.}}{\sim} L$, and $b_1, \dots, b_q \overset{\text{i.i.d.}}{\sim} R$. The constraint in Eq. (1) implies the RHS is at least $1/2$. Note now that since the RHS is convex in any b_q, we can preserve the objective above and increase the RHS when b_q is drawn from a Bernoulli distribution with mean $\mathbb{E}[R]$. We next absorb the mass at 0 in R into the distribution L; call the new distributions $\hat{L}, \hat{R}$. Therefore, $\hat{R}$ is the deterministic value 1, and $\Pr[\hat{R}] = 1-\alpha$. Then, the objective is

$$(1-\alpha) - \alpha \mathbb{E}[\hat{L}].$$

Since each $b_q = 1$ now, the constraint on $p_k(W,X)$ becomes

$$p_k(W,X) \le \sum_{\ell=1}^{k} \binom{k}{\ell} \alpha^{\ell} (1-\alpha)^{k-\ell} \mathbb{E}\left[\frac{\sum_{r=1}^{\ell} a_r}{\sum_{r=1}^{\ell} a_r + k - \ell}\right],$$

where the expectation is now over $a_1, \dots, a_k \overset{\text{i.i.d.}}{\sim} \hat{L}$. Noting that the RHS is concave in a_r, by Jensen's inequality, we have

$$p_k(W,X) \le \sum_{\ell=1}^{k} \binom{k}{\ell} \alpha^{\ell} (1-\alpha)^{k-\ell} \cdot \frac{\ell \cdot \mathbb{E}[\hat{L}]}{\ell \cdot \mathbb{E}[\hat{L}] + k - \ell}.$$

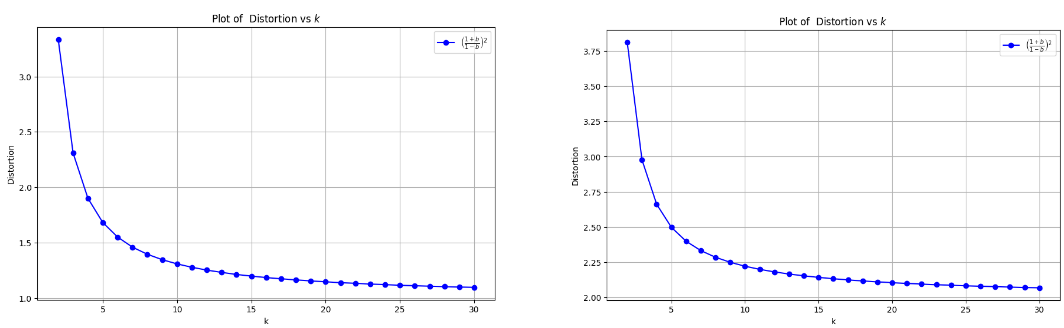
Therefore, using $\omega = \mathbb{E}[\hat{L}]$ and observing that $\omega, \alpha \in [0,1]$, Eq. (1) can be rewritten as:

$$\zeta_k := \max_{\omega,\alpha\in[0,1]} (1-\alpha) - \alpha \cdot \omega \qquad \text{s.t.} \qquad \sum_{\ell=1}^{k} \binom{k}{\ell} \alpha^{\ell}(1-\alpha)^{k-\ell} \cdot \left[\frac{\ell \cdot \omega}{\ell \cdot \omega + k - \ell}\right] \geq \frac{1}{2}. \tag{16}$$

For any given α, the LHS of the constraint is concave and increasing in ω, while the objective is decreasing in it, so it can be optimized by a binary search over ω to find the smallest ω for which the constraint is satisfied. We then run a parametric search over α in increments of 10^{-3} to find an approximate global optimum. By Theorem 2.2, the distortion is at most $\left(\frac{1+\zeta_k}{1-\zeta_k}\right)^2$. We numerically compute this for $2 \leq k \leq 30$, and plot it in Fig. 1a, and show in the second column in Table 2. This yields the following theorem.

Theorem 4.1. *For the Copeland rule applied to the random choice deliberation model with group size k, the distortion is at most 3.34 for $k = 2$, at most 2.31 for $k = 3$, and at most 1.90 for $k = 4$.*

Note that the upper bound for $k = 2$ is significantly lower than the lower bound of 4.414 for the averaging model in Theorem 3.3. Note further that the distortion for $k = 3$ is well below the distortion of 2.74 of the best randomized social choice rule without deliberation, and that for $k = 4$ is below the lower bound of 2.11 on metric distortion without deliberation.



(a) Random Choice Model. (b) Generalized Random Choice, $g(x) = \sqrt{x}$.

Figure 1: Distortion of the Random Choice model as a function of the group size k, for $k \in [2, 30]$.

Lower Bound for any Social Choice Rule. Let ζ_k be the optimal value to Eq. (16) for groups of size k. Note that this corresponds to the expectation of a valid distribution over $\{-\omega, 1\}$, where $\Pr[-\omega] = \alpha$. By Theorem 2.3, this implies a distortion of $\frac{1+\zeta_k}{1-\zeta_k}$ for any deterministic social choice rule, and $\frac{1}{1-\zeta_k}$ for any randomized rule. These values are shown in the third and fourth columns in Table 2, and show that our upper bounds above are reasonably tight.

4.2 Asymptotic Behavior of Distortion in Group Size k

We next show that the group size needed for Copeland to achieve a distortion of $1 + \varepsilon$ is $\tilde{O}(1/\varepsilon^2)$, independent of the number of alternatives.

Theorem 4.2. *For any $\varepsilon > 0$, with a group size of $k = O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}\right)$, the distortion of the Copeland rule in the random choice deliberation model is at most $1 + \varepsilon$.*

Proof. Consider the optimization problem in Eq. (16). We can view it as finding a distribution D that is $-\omega$ with probability α and 1 with probability $1-\alpha$, from which k samples are drawn. If $\alpha < 1/2$, the constraint cannot be satisfied for any $\omega \in [0,1]$, so that in the optimal solution $\alpha \geq 1/2$. For any $\delta > 0$, by Chernoff bounds,

$$\Pr[\ell \geq (1+\delta)k\alpha] \leq e^{-k\alpha\delta^2/2} \leq e^{-k\delta^2/4}.$$

If this event does not happen, then for $\ell^* = (1+\delta)k\alpha$, we have

$$\frac{\ell \cdot \omega}{\ell \cdot \omega + k - \ell} \leq \frac{\ell^* \cdot \omega}{\ell^* \cdot \omega + k - \ell^*},$$

since the LHS is monotonically increasing in ℓ. Letting $\beta = (1+\delta)\alpha$, by the law of total probability, the constraint therefore implies

$$\frac{\beta \cdot \omega}{\beta \cdot \omega + 1 - \beta} + e^{-k\delta^2/4} \geq \frac{1}{2}.$$

Let $k = \frac{4}{\delta^2}\log(2/\delta)$. Then, we have

$$\frac{\beta \cdot \omega}{\beta \cdot \omega + 1 - \beta} \geq \frac{1-\delta}{2} \;\Rightarrow\; \frac{(1+\delta)^2}{1-\delta} \cdot \alpha \cdot \omega \geq 1 - \alpha - \alpha \cdot \delta \;\Rightarrow\; 1 - \alpha - \alpha \cdot \omega = O(\delta).$$

Choosing $\varepsilon = c \cdot \delta$ for a suitable constant c, this shows $\zeta_k = O(\varepsilon)$. Plugging this into Theorem 2.2 completes the proof. □

4.3 Sample Complexity

As in Theorem 3.9, we bound the number of samples needed to estimate all $p_k(W,X)$ to within additive ε. The key difference now is that the Random Choice model is only defined for pairs of alternatives. Consider the complete graph K_m on the alternatives, and split the set of edges into $m-1$ matchings. Given a group of voters, we pick one of the $m-1$ matchings. This group of voters deliberates over the m pairs of alternatives in this matching, outputting one alternative for each pair using the Random choice model. Fixing a matching, if we use this matching for $O\left(\frac{\log(m/\delta)}{\varepsilon^2}\right)$ groups, then for every edge (W,X) in the matching, we can estimate $p_k(W,X)$ to an additive error of ε with probability $1-\frac{\delta}{m}$. Repeating this for all $m-1$ matchings, all of the quantities $p_k(W,X)$ can be estimated with this error. As in Theorem 3.9, with high probability, the alternative W chosen by the Copeland rule run on the tournament graph over samples is such that all alternatives X, either $p_k(W,X) \geq 1/2-\varepsilon$, or there exists Y such that $p_k(W,Y) \geq 1/2-\varepsilon$ and $p_k(Y,X) \geq 1/2-\varepsilon$. Since ζ_k in Eq. (16) is smooth in the RHS of the constraint, the optimum shifts by at most an additive $O(\varepsilon)$. This finally yields the following theorem.

Theorem 4.3. *Let d_k denote the distortion of the Copeland rule with groups of size k assuming each $p_k(W,X)$ can be estimated exactly. Then, $O\left(\frac{m\log(m/\delta)}{\varepsilon^2}\right)$ randomly chosen groups of size k suffice for distortion $d_k+\varepsilon$ with probability $1-\delta$.*

As mentioned after Theorem 3.9, our sampling bounds with constant size groups are not implied by similar bounds for the 1-median problem, and crucially require voters to output rankings beyond their favorite (or median) alternative.

4.4 Generalized Random Choice

We now consider two generalizations to the model that are amenable to the same analysis technique as in Section 4.1. We present the model and the program in each case, and except for one canonical instance, we omit the easy computation of the numerical bounds for distortion.

Concave Bias. In the first generalization, the outcome of deliberation is less dominated by extremely biased voters. The model now has a concave, non-decreasing function g, with $g(0) = 0$ and $g(1) = 1$. For any pair of outcomes W, X, given a multiset S of voter locations of size k, the probability the outcome of deliberation is the favorite outcome of $i \in S$ is proportional to $g(|\mathcal{B}_i(W, X)|)$. The Random Choice model considered so far had $g(x) = x$. Note that as g becomes more concave (away from linear), the model favors a more uniformly random voter in the deliberating group, as opposed to a more biased voter.

For any such g, the function $\frac{g(x)}{g(x)+c}$ is also concave and non-decreasing in x. We can re-derive the optimization problem in Eq. (16) by analogously applying Jensen's inequality. This yields

$$\zeta_k := \max_{\omega,\alpha\in[0,1]} (1-\alpha) - \alpha \cdot \omega \qquad \text{s.t.} \qquad \mathbb{E}_{\ell\sim\texttt{Bernoulli}(k,\alpha)} \left[\frac{\ell \cdot g(\omega)}{\ell \cdot g(\omega) + k - \ell} \right] \geq \frac{1}{2}. \tag{17}$$

for a distortion of at most $\left(\frac{1+\zeta_k}{1-\zeta_k}\right)^2$ by Theorem 2.2. It is now easy to compute the distortion for any fixed function g. We plot the distortion as a function of k for $g(x) = \sqrt{x}$ in Fig. 1b. Observe that the distortion is at most 2.98 for $k = 3$, and asymptotically approaches 2 as $k \to \infty$.

This shows that the asymptotic distortion bound of 1 in Theorem 4.2 is specific to the model with $g(x) = x$, and making $g(x)$ more concave leads to asymptotically larger distortion. This is intuitive, since if $g(x) = 1$, then the model reduces to random dictatorship, which has distortion 3.

Opinion Change of Voters. In the second generalization, we model a voter in the group S as changing their opinion. Let $S_1 \subseteq S$ denote the set of voters who perfer W to X, and let $S_2 \subseteq S$ denote those that prefer X to W. Let $A = \sum_{i\in S_1} |\mathcal{B}_i(W, X)|$ denote the total normalized bias of voters preferring W, and let $B = \sum_{i\in S_2} |\mathcal{B}_i(W, X)|$ be that for X.

There is a parameter $\beta \in [0, 1]$. During deliberation, suppose every voter in S_1 independently changes their opinion to X with probability $\beta \cdot \frac{B}{A+B}$, and similarly, every voter in S_2 changes their opinion to W with probability $\beta \cdot \frac{A}{A+B}$. Subsequently, the opinion of a randomly chosen voter in the group is implemented. It can be checked that

$$p_k(W, X) = \beta \cdot \frac{A}{A+B} + (1-\beta) \cdot \frac{|S_1|}{|S|}.$$

Note that if $\beta = 1$, this is exactly the random choice model, whereas if $\beta = 0$, this is simply random dictatorship. The parameter β captures the extent of opinion change. For any $\beta \in [0, 1]$, we can upper bound the distortion using the same technique as in Section 4.1. Indeed, Eq. (16) becomes:

$$\zeta_k := \max_{\omega,\alpha\in[0,1]} (1-\alpha) - \alpha \cdot \omega \qquad \text{s.t.} \qquad \beta \cdot \sum_{\ell=1}^{k} \binom{k}{\ell} \alpha^\ell (1-\alpha)^{k-\ell} \cdot \left[\frac{\ell \cdot \omega}{\ell \cdot \omega + k - \ell} \right] + (1-\beta) \cdot \alpha \geq \frac{1}{2},$$

which can be optimized as in Section 4.1 for any fixed β. As with the previous generalization, the distortion bound will be asymptotically larger than 1 if $\beta < 1$.

5 Open Questions

We now list some open questions. At the technical level, the main question is to close the gap between the lower and upper bounds in Tables 1 and 2. An intriguing question for $k = 2$ is the following: If every pair of voters deliberates and outputs a ranking over alternatives that is consistent with the sum of their distances to the alternatives, is there a deterministic social choice rule over these rankings that beats distortion 3? The lower bound example in Theorem 3.3 can be extended to weighted tournament rules [MW19, Kem20], and shows these rules are insufficient. Further, rules like plurality veto [KK22] that output the favorite alternative of some voter are insufficient by Example 1. Finding a rule for $k = 2$ that admits to tractable analysis, yet breaks the bound of 3 is an interesting question.

More generally, tightening our bounds will require analysis of rules beyond tournaments, which poses analytic challenges. Even for tournament rules like Copeland, it would be interesting to bypass the use of non-convex optimization tools in deriving the results in Section 3, which will enable extensions to larger group sizes. Indeed, our conjecture is that for all $k \geq 2$, the optimal solution to the non-convex program in Section 3 has the same binary support as that of the lower bound instances in Theorem 3.2. It would also be interesting to develop a tight distortion bound for Copeland for $k = 3$, improving Theorem 3.7.

At the modeling level, we can consider richer models of how voters randomize between outcomes. For instance, the work of [GS25] proposes a randomized voting model where $\Pr[W \succ_i X] = g(r)$, where $r = d(i, W)/d(i, X)$, and $g(r)$ is a bounded, increasing function, such as $g(r) = \frac{r^2}{1+r^2}$. In this model, they show improved distortion bounds without deliberation, and it would be interesting to analyze extensions to deliberation. Further, in our model, we have assumed the groups are chosen by random sampling. In practice, the sampling could be stratified based on voter features, or their prior ranking of the alternatives. It would be interesting to extend the model in Section 4 to handle such generalizations, and find the optimal stratification. Further, individuals often enter into deliberation with goals beyond simply persuading others, for instance, they may seek to understand different viewpoints, or strive for the common good [AFE$^+$24]. It would be interesting to model these aspects formally.

Finally, there are aspects of real-world deliberation that we have not modeled. For instance, party affiliation and signaling significantly impacts opinion formation and social learning (see [GBC18] and citations therein), since some people take their cues from a higher authority like party, or could defer to the partisan group, especially when they know little about a topic. Similarly, some individuals are better at persuasion than others (regardless of the opinions they hold), and it would be interesting to add individual capability into the model.

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References

[AAZ19] Ben Abramowitz, Elliot Anshelevich, and Wennan Zhu. Awareness of voter passion greatly improves the distortion of metric social choice. In Ioannis Caragiannis, Vahab Mirrokni, and Evdokia Nikolova, editors, *Web and Internet Economics*, pages 3–16, Cham, 2019. Springer International Publishing.

[ABB+23] Lisa P. Argyle, Christopher A Bail, E. Busby, Joshua R Gubler, Thomas Howe, Christopher Rytting, Taylor Sorensen, and David Wingate. Leveraging AI for democratic discourse: Chat interventions can improve online political conversations at scale. *Proceedings of the National Academy of Sciences of the United States of America*, 120, 2023.

[ABE+18] Elliot Anshelevich, Onkar Bhardwaj, Edith Elkind, John Postl, and Piotr Skowron. Approximating optimal social choice under metric preferences. *Artif. Intell.*, 264:27–51, 2018.

[AFE+24] Joshua Ashkinaze, Emily Fry, Narendra Edara, Eric Gilbert, and Ceren Budak. Plurals: A system for guiding LLMs via simulated social ensembles, 2024.

[AFRV22] Elliot Anshelevich, Aris Filos-Ratsikas, and Alexandros A. Voudouris. The distortion of distributed metric social choice. *Artificial Intelligence*, 308:103713, 2022.

[AFSV21] Elliot Anshelevich, Aris Filos-Ratsikas, Nisarg Shah, and Alexandros A. Voudouris. Distortion in social choice problems: An annotated reading list. *SIGecom Exch.*, 19(1):12–14, 2021.

[AP17] Elliot Anshelevich and John Postl. Randomized social choice functions under metric preferences. *J. Artif. Intell. Res.*, 58:797–827, 2017.

[AS04] Alexander A. Ageev and Maxim Sviridenko. Pipage rounding: A new method of constructing algorithms with proven performance guarantee. *Journal of Combinatorial Optimization*, 8:307–328, 2004.

[Asc55] Solomon E. Asch. Opinions and social pressure. *Scientific American*, 193(5):31–35, 1955.

[Axt18] Paul Axtell. The most productive meetings have fewer than 8 people. *Harvard Business Review*, June 2018.

[BCS+24] Michiel A. Bakker, Martin J. Chadwick, Hannah R. Sheahan, Michael Henry Tessler, Lucy Campbell-Gillingham, Jan Balaguer, Nat McAleese, Amelia Glaese, John Aslanides, Matthew M. Botvinick, and Christopher Summerfield. Fine-tuning language models to find agreement among humans with diverse preferences. In *Proceedings of the 36th International Conference on Neural Information Processing Systems*, NeurIPS '22, 2024.

[BLM13] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration Inequalities - A Nonasymptotic Theory of Independence.* Oxford University Press, 2013.

[BP05] Dimitris Bertsimas and Ioana Popescu. Optimal inequalities in probability theory: A convex optimization approach. *SIAM Journal on Optimization*, 15(3):780–804, 2005.

[CMP24] Ioannis Caragiannis, Evi Micha, and Jannik Peters. Can a few decide for many? The metric distortion of sortition. In *Forty-first International Conference on Machine Learning, ICML 2024, Vienna, Austria, July 21-27, 2024.* OpenReview.net, 2024.

[CR86] Jeesen Chen and Herman Rubin. Bounds for the difference between median and mean of Gamma and Poisson distributions. *Statistics & Probability Letters*, 4(6):281–283, 1986.

[CR22] Moses Charikar and Prasanna Ramakrishnan. Metric distortion bounds for randomized social choice. In Joseph (Seffi) Naor and Niv Buchbinder, editors, *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*, pages 2986–3004. SIAM, 2022.

[CWRW24] Moses Charikar, Kangning Wang, Prasanna Ramakrishnan, and Hongxun Wu. Breaking the metric voting distortion barrier. In David P. Woodruff, editor, *Proceedings of the 2024 ACM-SIAM Symposium on Discrete Algorithms, SODA 2024, Alexandria, VA, USA, January 7-10, 2024*, pages 1621–1640. SIAM, 2024.

[DBM11] John S Dryzek, André Bächtiger, and Karolina Milewicz. Toward a deliberative global citizens' assembly. *Global Policy*, 2(1):33–42, 2011.

[DeG74] Morris H DeGroot. Reaching a consensus. *Journal of the American Statistical association*, 69(345):118–121, 1974.

[DG55] Morton Deutsch and Harold Benjamin Gerard. A study of normative and informational social influences upon individual judgement. *Journal of abnormal psychology*, 51 3:629–36, 1955.

[Fei04] Uriel Feige. On sums of independent random variables with unbounded variance, and estimating the average degree in a graph. In *Proceedings of the Thirty-Sixth Annual ACM Symposium on Theory of Computing*, STOC '04, page 594–603, New York, NY, USA, 2004. Association for Computing Machinery.

[Fel71] William Feller. *An introduction to probability theory and its applications.* New York: John Wiley,, 1971.

[FFM20] Brandon Fain, William Fan, and Kamesh Munagala. Concentration of distortion: The value of extra voters in randomized social choice. In Christian Bessiere, editor, *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020*, pages 110–116. ijcai.org, 2020.

[FGMP19] Brandon Fain, Ashish Goel, Kamesh Munagala, and Nina Prabhu. Random dictators with a random referee: Constant sample complexity mechanisms for social choice. In *The Thirty-Third AAAI Conference on Artificial Intelligence, AAAI 2019, Honolulu, Hawaii, USA, January 27 - February 1, 2019*, pages 1893–1900. AAAI Press, 2019.

[FGMS17] Brandon Fain, Ashish Goel, Kamesh Munagala, and Sukolsak Sakshuwong. Sequential deliberation for social choice. In Nikhil R. Devanur and Pinyan Lu, editors, *Web and Internet Economics - 13th International Conference, WINE 2017, Bangalore, India, December 17-20, 2017, Proceedings*, volume 10660 of *Lecture Notes in Computer Science*, pages 177–190. Springer, 2017.

[Fis91] J.S. Fishkin. *Democracy and Deliberation: New Directions for Democratic Reform.* Yale University Press, 1991.

[FPW23] Bailey Flanigan, Ariel D. Procaccia, and Sven Wang. Distortion under public-spirited voting. In Kevin Leyton-Brown, Jason D. Hartline, and Larry Samuelson, editors, *Proceedings of the 24th ACM Conference on Economics and Computation, EC 2023, London, United Kingdom, July 9-12, 2023*, page 700. ACM, 2023.

[GBC18] Douglas Guilbeault, Joshua Becker, and Damon Centola. Social learning and partisan bias in the interpretation of climate trends. *Proceedings of the National Academy of Sciences*, 115(39):9714–9719, 2018.

[GHS20] Vasilis Gkatzelis, Daniel Halpern, and Nisarg Shah. Resolving the optimal metric distortion conjecture. In Sandy Irani, editor, *61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020*, pages 1427–1438. IEEE, 2020.

[GL16] Ashish Goel and David T. Lee. Towards large-scale deliberative decision-making: Small groups and the importance of triads. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, EC '16, page 287–303, New York, NY, USA, 2016. Association for Computing Machinery.

[GS25] Mohak Goyal and Sahasrajit Sarmasarkar. Metric distortion under probabilistic voting, 2025.

[HZZ10] Simai He, Jiawei Zhang, and Shuzhong Zhang. Bounding probability of small deviation: A fourth moment approach. *Mathematics of Operations Research*, 35(1):208–232, 2010.

[IL18] Sean Ingham and Ines Levin. Can deliberative minipublics influence public opinion? Theory and experimental evidence. *Political Research Quarterly*, 71(3):654–667, 2018.

[KB80] R. Kaas and J.M. Buhrman. Mean, median and mode in binomial distributions. *Statistica Neerlandica*, 34(1):13–18, 1980.

[Kem20] David Kempe. An analysis framework for metric voting based on LP duality. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34(02):2079–2086, Apr. 2020.

[KEO+20] Soomin Kim, Jinsu Eun, Changhoon Oh, Bongwon Suh, and Joonhwan Lee. Bot in the bunch: Facilitating group chat discussion by improving efficiency and participation with a chatbot. In *Proceedings of the 2020 CHI Conference on Human Factors in Computing Systems*, CHI '20, page 1–13, New York, NY, USA, 2020. Association for Computing Machinery.

[KK22] Fatih Erdem Kizilkaya and David Kempe. Plurality veto: A simple voting rule achieving optimal metric distortion. In Luc De Raedt, editor, *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI 2022, Vienna, Austria, 23-29 July 2022*, pages 349–355. ijcai.org, 2022.

[KS18] Aida Khajavirad and Nikolaos Sahinidis. A hybrid LP/NLP paradigm for global optimization relaxations. *Mathematical Programming Computation*, 10, May 2018.

[Mei07] Adam Meirowitz. In defense of exclusionary deliberation: Communication and voting with private beliefs and values. *Journal of Theoretical Politics*, 19(3):301–327, 2007.

[Mil77] Nicholas R. Miller. Graph-theoretical approaches to the theory of voting. *American Journal of Political Science*, 21(4):769–803, 1977.

[mis] Mismatch. `https://mismatch.org/`.

[MU17] Michael Mitzenmacher and Eli Upfal. *Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis.* Cambridge University Press, USA, 2nd edition, 2017.

[MW19] Kamesh Munagala and Kangning Wang. Improved metric distortion for deterministic social choice rules. In Anna R. Karlin, Nicole Immorlica, and Ramesh Johari, editors, *Proceedings of the 2019 ACM Conference on Economics and Computation, EC 2019, Phoenix, AZ, USA, June 24-28, 2019*, pages 245–262. ACM, 2019.

[myj] Myjunto. `https://myjunto.app/home/`.

[PNC06] Vincent Price, Lilach Nir, and Joseph N. Cappella. Normative and informational influences in online political discussions. *Communication Theory*, 16(1):47–74, 03 2006.

[PR06] Ariel D. Procaccia and Jeffrey S. Rosenschein. The distortion of cardinal preferences in voting. In *Proceedings of the 10th International Conference on Cooperative Information Agents*, CIA'06, page 317–331, Berlin, Heidelberg, 2006. Springer-Verlag.

[Sah96] Nikolaos V. Sahinidis. BARON: A general purpose global optimization software package. *Journal of Global Optimization*, 8:201–205, 1996.

[Sie01] Alan Siegel. Median bounds and their application. *J. Algorithms*, 38(1):184–236, jan 2001.

[sta] Online deliberation platform. `https://stanforddeliberate.org/`.

[Szp10] George G. Szpiro. *Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present.* Princeton University Press, 2010.

[vTa] vTaiwan: Rethinking democracy. `https://info.vtaiwan.tw/`.

[WP08] Mark E Warren and Hilary Pearse. Designing deliberative democracy: The British Columbia citizens' assembly. 2008.