# Mon 8/24

Today:

(1) Example of matrix multiplication.

(2) Inverse of a square matrix.

(3) Rank of a matrix

**Example.** Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$. Find $AB$.

**Solution.**

$$\begin{array}{cc} & \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} \\ \begin{bmatrix} 13 \\ 24 \end{bmatrix} & \boxed{\begin{bmatrix} 5+18 & 7+24 \\ 10+24 & 14+32 \end{bmatrix}} = \begin{bmatrix} 23 & 31 \\ 34 & 46 \end{bmatrix} \end{array}$$

**Definition.** If a matrix $A$ is **invertible** then there exists matrix $B$ such that $AB = I$.

**Problem 1.** Does $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have an inverse?

**Solution.** Suppose yes and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+2b_{21} & b_{12}+2b_{22} \\ 2b_{11}+4b_{21} & 2b_{12}+4b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is absurd.

**Remark.** Another explanation of why $A$ is not invertible: $Ax = 0$ has a nontrivial solution. On the other hand, if $A$ is invertible then there exists $A^{-1}$, and $A^{-1}Ax = A^{-1}0 = 0$ and $A^{-1}Ax = Ix = 0 \implies x = 0$.

**Example.** Compute $A^{-1}$ where $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

**Solution.** Note that $A^{-1}\left[A \mid I\right] = \left[A^{-1}A \mid A^{-1}I\right] = \left[I \mid A^{-1}\right]$. Now if we do the same thing to $A$ :

$$A_{\text{aug}} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 \end{bmatrix} \implies$$

$$\begin{bmatrix} 1 & 0 & -2 & 1.5 \\ 0 & 2 & 2 & -1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & -2 & -1.5 \\ 0 & 1 & 1 & -0.5 \end{bmatrix} \implies A^{-1} = \begin{bmatrix} -2 & 1.5 \\ 1 & 0.5 \end{bmatrix}$$

# Wed 8/26

Today:

(1) $A$ and inverse of $AB$.

(2) LU decomposition of $A$.

**Theorem.** $(AB)^T = B^T A^T$.

**Remark.** In general $AB \neq BA$, i.e., matrix multiplication is not commutative.

Now suppose $A = \begin{bmatrix} R_1(A) \\ R_2(A) \end{bmatrix}$ and $B = \begin{bmatrix} C_1(B) & C_2(B) \end{bmatrix}$. Then

$$(AB)^T = \begin{bmatrix} R_1(A)\cdot C_1(B) & R_1(A)\cdot C_2(B) \\ R_2(A)\cdot C_1(B) & R_2(A)\cdot C_2(B) \end{bmatrix}^T = \begin{bmatrix} R_1(A)\cdot C_1(B) & R_2(A)\cdot C_1(B) \\ R_1(A)\cdot C_2(B) & R_2(A)\cdot C_2(B) \end{bmatrix}.$$

On the other hand,

$$B^T A^T = \begin{bmatrix} C_1(B) \\ C_2(B) \end{bmatrix} \begin{bmatrix} R_1(A) & R_2(A) \end{bmatrix} = \begin{bmatrix} R_1(A)\cdot C_1(B) & R_2(A)\cdot C_1(B) \\ R_1(A)\cdot C_2(B) & R_2(A)\cdot C_2(B) \end{bmatrix} i$$

from which we see indeed $(AB)^T = B^T A^T$.

Now let's find the inverse of $A^T$ :

$$AA^{-1} = I \implies (AA^{-1})^T = I = (A^{-1})^T A^T$$

in other words "the inverse of transpose is the same as the transpose of inverse".

## LU Decomposition

**Example.** $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

**Solution.**

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$$

because we want to subtract 2 times row 1 from row 2 and make it the new row 2. Namely, adding original row 2 by $(-2)$ times row 1. Hence $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. Then $E_{21}A = U \implies A = E_{21}^{-1}U \implies L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, namely "adding the previously removed 2 times row 1 back to row 2.

**Remark.** L is unit lower triangular: 1's along the diagonal and lower triangular. The entry $L_{21}$ is the multiplier used to do row operations.

If there are more than one elimination matrices, for example, $E_{32}E_{31}E_{21}A = U$, then
$A = (E_{32}E_{31}E_{21})^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$.

We can also make it $A = LDU$, where $LU$ is a diagonal matrix that factors out all diagonal entries of $U$ and make it a unit upper triangular.

## Fri 8/29

Today:

(1) LU factorization

(2) Permutation matrices

(3) Vector spaces