Mon 8/31, 4.2&4.3, Vector Spaces

Definition. A **vector space** is a set that is closed under two operations that we call addition and scalar multiplication.

Example. $\mathbb{R}^n$, the set of all 2×2 matrices, the set of polynomials of deg $\geqslant 2$ with real coefficients, the set of all continuous functions $f : [a, b] \to \mathbb{R}$.

Example. Not a vector space: $\mathbb{R}^{\geqslant 0} \times \mathbb{R}^{\geqslant 0}$: scalar multiplication by a negative number.

Remark. Ten properties of vector spaces.

Definition. A **subspace** is a vector space within another vector space.

Example. Any line in $\mathbb{R}^2$ that passes through the origin is a subspace of $\mathbb{R}^2$. The singleton of zero element is a *trivial* subspace of any vector space. In $\mathbb{R}^2$ it would be $(0, 0)$, in $M_{2\times 2}(\mathbb{R})$ it would be $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, etc. The other *trivial* subspace is the vector space itself.

Example. Suppose $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ where $C_1(A) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $C_2(A) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. The set of vectors spanned by $C_1(A)$ and $C_2(A)$ is a vector space (subspace) called the **column space** of A, denoted as $C(A)$.

Theorem. $A\mathbf{x} = \mathbf{b}$ has solutions if and only if $\mathbf{b} \in C(A)$.

Definition. The following are the fundamental subspaces:

(1) **Column space**: $C(A)$, the vector space spanned by column vectors of A, aka range(A).

(2) **Nullspace**: $N(A)$, the set of solutions to $A\mathbf{x} = 0$.

(3) **Row space**: $C(A^T)$, the vector space spanned by row vectors of A.

(4) **Left nullspace**: $N(A^T)$, the set of solutions to $A^T\mathbf{y} = 0$.

Wed 9/2

Today:

(1) Subspaces, $N(A)$, $C(A) = \text{range}(A)$

(2) Spanning sets

(3) Linear independence

Example. Let $A = \begin{bmatrix} 1 & -4 & 6 \\ -3 & 10 & -10 \end{bmatrix}$

(1) Show that $N(A)$ is a subspace of $\mathbb{R}^3$.

(2) Find that subspace.

Solution.

(1) Suppose $\mathbf{u}, \mathbf{v} \in N(A)$. It follows that $A\mathbf{u} = A\mathbf{v} = 0 = A(\mathbf{u} + \mathbf{v})$. Therefore $N(A)$ is closed under addition. Likewise suppose $A\mathbf{u} = 0$, then $A(k\mathbf{u}) = k0 = 0$ and $N(A)$ is closed under scalar multiplication.

(2)
$$\begin{bmatrix} 1 & -4 & 6 & 0 \\ -3 & 10 & -10 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -4 & 6 & 0 \\ 0 & -2 & 8 & 0 \end{bmatrix}$$

Since x_3 does not correspond to pivots, the third column is a free column, and we set x_3 as the free variable, whereas x_1, x_2 are basic variables. Set $x_3 = t$. Then we have $x_2 = 4t$ and $x_2 = 10t$. Then the nullspace consists of vectors of the form $c\begin{bmatrix} 10 & 4 & 1 \end{bmatrix}^T$.

When there are > 1 free variables, to get a special solution, set one free variable to 1 then set every other free variable to 0. Then do the same for every other free variable.

Definition. If every vector v in the vector space $\mathcal{V}$ can be written as a linear combination of the vectors $v_1, v_2, \ldots, v_k$, we say that $\mathcal{V}$ is spanned by (or generated by) $\{v_1, v_2, \ldots, v_k\}$.

Example. Do $v_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ span $\mathbb{R}^2$?

Solution. Take $\begin{bmatrix} b_1 & b_2 \end{bmatrix}^T$, an arbitrary vector, in R^2. We will show that $\begin{bmatrix} v_1 & v_2 \end{bmatrix}\mathbf{x} = \mathbf{b}$ has a solution.

$$\begin{bmatrix} 1 & 2 & b_1 \\ 1 & 3 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_2 - b_1 \end{bmatrix}$$

Therefore $\begin{bmatrix} 3b_1 - 2b_2 \\ b_2 - b_1 \end{bmatrix}$ is always a solution to $\begin{bmatrix} v_1 & v_2 \end{bmatrix}\mathbf{x} = \mathbf{b}$.

Example. Suppose $A = \begin{bmatrix} -1 & 5 & 3 \\ 2 & -10 & 6 \end{bmatrix}$. Find $N(A)$.

Solution. REF provides $A \sim \begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Then we have two special solutions: $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Therefore $N(A)$ is a subspace in $\mathbb{R}^3$ spanned by $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Definition. We say that the vectors $v_1, v_2, \dots, v_n$ are **linearly independent** if and only if

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

only when $c_1 = c_2 = \dots = c_n = 0$.

Definition. The functions $f_1(x), f_2(x), \dots, f_n(x) : \mathbb{R} \to \mathbb{R}$ are linearly independent on $[a, b]$ if and only if the linear combinations of them equal to 0 for all $x \in [a, b]$ only when all coefficients are 0.

Fri 9/4

Today:

(1) linear independent functions

(2) bases and dimension of a vector space

(3) row space and column space if possible

Recall that if we have $f_1(x), f_2(x)$ both from $I = [a, b]$ to $\mathbb{R}$, we say they are linearly independent if $c_1 f_1(x) + c_2 f_2(x) = 0$ only when $c_1 = c_2 = 0$. If $f_1(x), f_2(x)$ are dependent over the interval, then

$$-c_1 f_1(x) = c_2 f_2(x) \implies f_2(x) = -\frac{c_1}{c_2} f_1(x) \text{ (a multiple of } f_1(x))$$

Of course we want a more systematic way to check if a set of functions are linearly independent or not... Introducing the Wronskian. First look at $f_1(x), f_2(x)$ again. Suppose they are differentiable over I. Then

$$c_1 f_1(x) + c_2 f_2(x) = 0$$
$$c_1 f_1'(x) + c_2 f_2'(x) = 0$$

and in matrix form

$$\begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that if the left 2×2 is invertible ($\det \neq 0$) then the only solution is $c_1 = c_2 = 0$, namely the two functions are independent.

Definition. For a $f_1(x), f_2(x), \dots, f_n(x) \in C^{n-1}[a, b]$, the **Wronskian** of these functions is

$$\mathcal{W}[f_1 f_2, \dots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem. If $f_1(x), f_2(x), \dots, fn(x) \in C^{n-1}[a, b]$ and $\mathcal{W}[\dots](x_0) \neq 0$ at some $x_0 \in [a, b]$, then the functions are linearly independent over the interval $[a, b]$

Remark. Even if the Wronskian is identically zero over an interval, we still cannot conclude if the functions are linearly dependent.

Definition. A set of vectors $\{v_1, \ldots, v_k\}$ form a **basis** for a vector space $\mathcal{V}$ if

(1) they are linearly independent and

(2) they span the vector space $\mathcal{V}$.

Definition. In $\mathbb{R}^n$, each vector in the **standard basis** has 1 for one component and 0 for every other component (and they together form a basis).

Wed 9/9

Previously:

(1) linear independence of vectors and functions, Wronskian

(2) "baby" example of determinant

(3) span and basis

Today:

(1) dimension of a vector space

(2) basis of a vector space

(3) some related results

Example. In $\mathbb{R}^2$, if for $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent, then $A = \begin{bmatrix} | & | \\ \mathbf{v_1} & \mathbf{v_2} \\ | & | \end{bmatrix}$ has nonzero solution, and $A\mathbf{x} = 0$ has only trivial solution. They span $\mathbb{R}^2$ if $A\mathbf{x} = \mathbf{b} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}^T$ always has a solution.

Definition. The number of vectors in a basis is called the **dimension** of the vector space.

Remark. Given a vector space, the number of vectors in a basis is unique. In other words, the dimension of the vector space is the same regardless of the choosing of basis.

Suppose we know that the dimension of a vector space $\mathcal{V}$ is n. If we are given n vectors, then the following are either both true or both false.

(1) they are linearly independent

(2) they span $\mathcal{V}$

Example. We know that $\dim(P_2(\mathbb{R})) = 3$. Question: is the set $\{1, x\}$ a basis of $P_2(\mathbb{R})$? How about the set $\{1, x, x^2 + 1, x^2\}$? How about $\{1, x, x^2\}$? How about $\{1, x, 2x\}$?

Solution. The first two are not because the cardinality of the sets $\neq 3$. The third is a basis because

(1) the cardinality of the set is 3, and

(2) $ax^2 + bx + c \in P_2(\mathbb{R}) = a \cdot x^2 + b \cdot x + c \cdot 1$, or that the Wronskian of $\{1, x, x^2\}$ is nonzero.

The last set is not a basis because it does not span the vector space with degree 2 terms.

Fri 9/11

Recently:

(1) Bases and dimension

(2) Row space and column space

(3) Rank-Nullity Theorem (FT of LA)

Recall that in $\mathbb{R}^2$, $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ would never be a basis because $\dim\left(\mathbb{R}^2\right) = 2 < |\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}| = 3$ and the vectors cannot be linearly independent. Likewise, any single vector cannot be a basis since $\mathbb{R}^2$ because it cannot possibly span $\mathbb{R}^2$. To be a basis, the vectors needs to be both linearly independent and be able to span the vector space.

Row Space & Column Space

Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$. Then $C(A^T)$, the rowspace of A (or the column space of A^T), is $\operatorname{span}\{R_1, R_2, R_3\}$.

Example. Find $C(A^T)$ for A above.

Solution. Note that by performing row operations, whatever the original row vectors span remains the same because we are doing linear combinations among the rows themselves. Then,

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 6 & -12 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $C(A^T) = \operatorname{span}\{\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}\}$.

Example. Find $C(A)$.

Solution. $C(A)$ can be computed by computing its equivalent $C((A^T)^T)$, the row space of A^T. Alternatively, we can proceed directly from the previous example. From the REF above we see that the pivots lie in the first two columns. Hence

$$C(A) = \text{span}\{C_1, C_2\} = \text{span}\{\begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}\}.$$

Note that $C(A)$ and $C(R)$ are in general different, but the pivots correspond.

Remark. $\dim((C(A)) = \dim\left(C(A^T)\right)$. It is no coincidence.

Theorem. To find $C(A)$, find the pivots of any REF of A. Then the original columns corresponding to the pivots form a basis for $C(A)$.

Proof. Immediate from the fact that $A\mathbf{x} = 0$ and $R\mathbf{x} = 0$ have the same solution sets. Then if the columns with pivots in R is a maximal linearly independent set, so are the corresponding columns in A. □

Theorem (Rank-Nullity)**.** If A is $m \times n$ with rank r, then $\dim(A) + \dim(N(A)) = r + (n - r) = n$.