1 Mon 8/17 (Ch 10.1/11.1)

$$f(x) : \underbrace{\mathbb{R}^n}_{\text{Domain}} \to \underbrace{\mathbb{R}^m}_{\text{Codomain / target space}}.$$

A function, by convention, is defined by not just the data of inputs and outputs but also the domain and codomain. That is, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}^+$ defined by $g(x) = x^2$ are different functions. In the first example, $\text{im}(f) = \mathbb{R}^+$ while the target space is the entire $\mathbb{R}$.

Level sets: similar to contour maps in some sense: the level set of $F : \mathbb{R}^2 \to \mathbb{R}$ at level c is similar to a contour line of height c :

$$L_c(F) = \{(x, y) \mid F(x, y) = c\}$$

Why are they used: one answer is that any time you define a set by saying (in $\mathbb{R}^3$ for example) "some constant equation involving the coordinates holds". Then you can turn the equation into a level set.

e.g. $2x - 3y + 5z = 7$ is an equation of a plane in $\mathbb{R}^3$.

For a level set, you'll always have n variable the domain $\mathbb{R}^n$, $x_1, x_2, \ldots, x_n$, and m variables in the codomain $\mathbb{R}^m$, $y_1, y_2, \ldots, y_m$. A level set is all $(x_1, x_2, \ldots, x_n)$ whose images are $\mathbf{c} = (c_1, c_2, \ldots, c_m)$.

2 Wed 8/19

Writing the equation of a sphere in $\mathbb{R}^3$ with radius r and center (h, k, ℓ) / writing this sphere as a level set (algebraicly):

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2$$

by the Pythagorian thm in $\mathbb{R}^3$ and the fact that $S^2 :=$ set of all points whose distance to (h, k, ℓ) is r.

By the same token, the distance between $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2 \ldots, y_n)$ in $\mathbb{R}^n$ is

$$d_{(\mathbb{R}^n)}(x, y) = \sum_{i=1}^{n} (x_i - y_i)^2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

We can also describe the sphere by the function

$$F : \mathbb{R}^3 \to \underbrace{\mathbb{R}}_{\text{Note the codomain is only } \mathbb{R} \text{ and image is } \mathbb{R}_{\geqslant 0}} \quad \text{defined by } F(x, y, z) := (x - h)^2 + (y - k)^2 + (z - \ell)^2$$

with the level set $S^2 = L_{r^2}(F)$

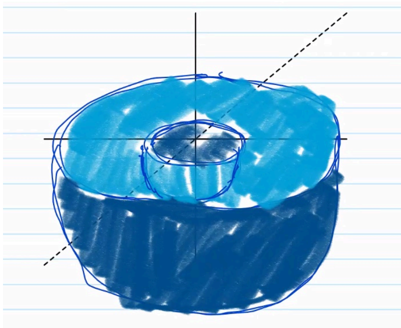
Level sets defines shapes using **equalities** but they can also use **inequalities**:

Example 2.1

Draw the set of (x, y, z) with $1 \leqslant x^2 + y^2 + z^2 \geqslant 4$ and $z \leqslant 0$.

Solution

The first equation gives a shell centered at the origin with inner radius 1 and outer radius 4. The second equation removes the top half of the shell. All diagrams are screenshots from Zoom class directly.



3 Thu 8/20 Discussion

Problem 1: 10.1.7

Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle?

(1) $P(3,-2,-3), Q(7,0,1), R(1,2,1)$

(2) $P(2,-1,0), Q(4,1,1), R(4,-5,4)$

Solution

(1) $\mid PQ \mid = \sqrt{4^2+2^2+4^2} = 6; \mid QR \mid = \sqrt{(-6)^2+2^2+0} = \sqrt{40} = 2\sqrt{10}; \mid PR \mid = \sqrt{(-2)^2+4^2+4^2} = 6$. A triangle with side lengths $6, 2\sqrt{10}, 6$ is an isoceles, but not right, triangle.

(2) $\mid PQ \mid = \sqrt{2^2+2^2+1^2} = 3; \mid QR \mid = \sqrt{0+(-6)^2+3^2} = \sqrt{45} = 3\sqrt{5}; \mid PR \mid = \sqrt{2^2+(-4)^2+4^2} = \sqrt{20} = 2\sqrt{5}$. Not isoceles, not right.

Problem 2: 10.1.19

Find equations of spheres with center $(2,-3,6)$ that touch the xy-plane.

Solution

Touching the xy plane and centered at $(2, -3, 6) \implies$ radius = 6. Therefore the equation is

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = 36.$$

Problem 3: 10.1.21-30

Describing regions in $\mathbb{R}^3$ represented by equations or equalities...Chill...

Problem 4: 10.2.12

Addition of vectors...Chill...

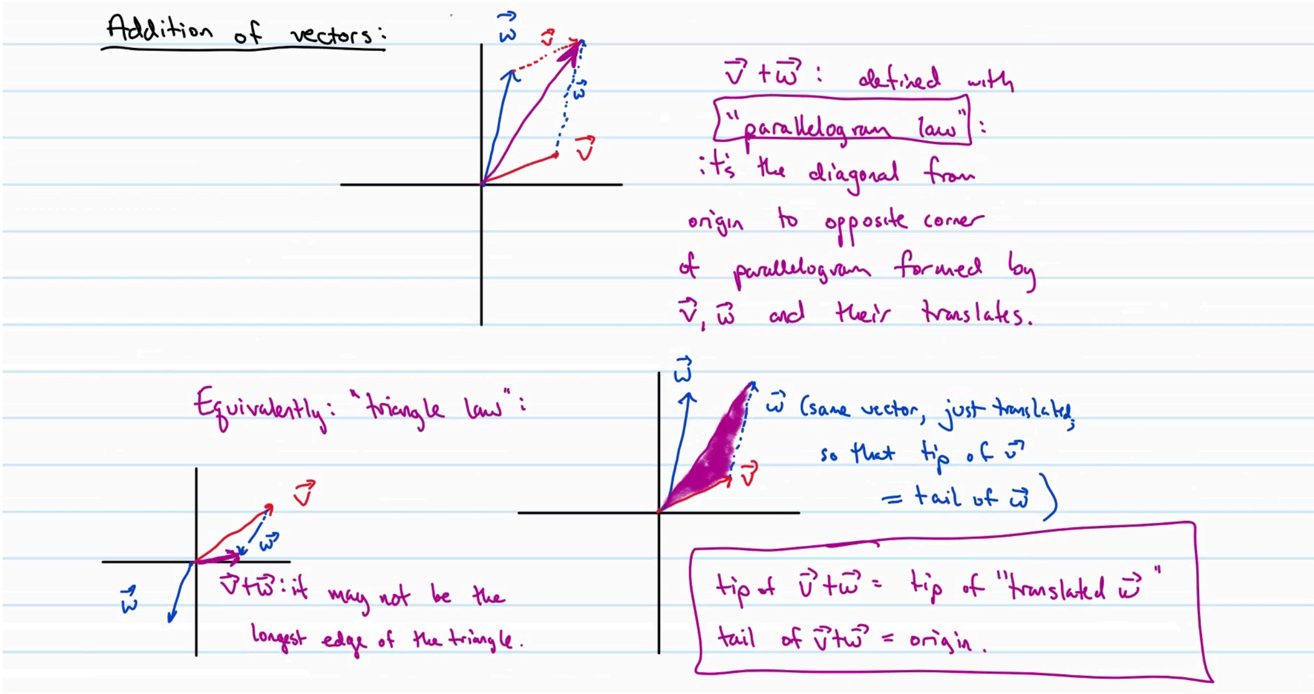
4 Fri 8/21 (Ch 10.2 Vectors)

Idea: **vectors** represent some quantities that have both magnitude and direction. (Contrast this with **scalar**). Notation: $\overrightarrow{AB}$, an arrow from tail to tip.

At undergrad level, $\mathbf{u} = \mathbf{v}$ if and only if they have the same magnitude and direction. *Vector bundle* at grad level requires additional conditions.

It's convenient to set the tail of an vector at the origin.

Addition:



Algebraic treatment of vectors:

Mtain idea: if we assume all vectors are *rooted at the origin*, then vectors are uniquely determined by their tip, a point in $\mathbb{R}^n$ (at least at undergrad level).

Two interpretations of $\mathbb{R}^n$:

(1) $\mathbb{R}^n$ as the geometric n-dimensional space. Focuses on points, $(x_1, x_2, \ldots, x_n)$.

(2) $\mathbb{R}^n$ as the set of all n-dimensional vectors. Focuses on vectors, $\langle x_1, x_2, \ldots, x_n \rangle$ or $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

(3) All representations represent ordered n-tuples.

	Algebraic representation	Visual representation
Addition	$\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$	"Parallelogram Law" & "Triangle Law"
Scalar Multiplication	$k \langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$	Scales length; direction unchanged

Basic properties: addition is commutative and distributive.

What to remember:

(1) Cannot add vectors from different dimensions: $\langle 3, 4 \rangle + \langle 1, 2, 4 \rangle$ is not defined.

(2) Multiplication is done by taking either dot product or cross product.

(3) If $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$ then the norm $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = \sqrt{\sum_{i=1}^n u_i^2}$.

(4) By convention, the "hat" `\hat` on top of $\hat{u}$ describes unit vector (of norm 1).

(5) Divide each component by norm to get a unit vector on the same direction.

5 Mon 8/24 10.2 & 10.3 Dot Products

Recall:

(1) In $\mathbb{R}^n$, there are n vectors that are especially important — the **standard basis vectors**: one component is 1 whereas all other components are 0.

(2) The standard basis vectors in $\mathbb{R}^3$ are

$$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle .$$

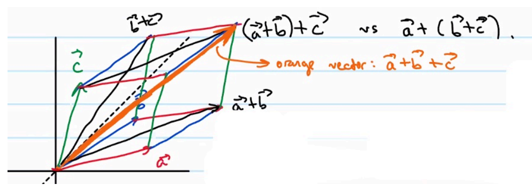
(3) These vectors are important because any $\langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$ can be written as $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Likewise for $\mathbb{R}^n$.

(4) Not all vectors vectors have direction (zero vector doesn't have direction), but all vectors have magnitude.

(5) If $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$, then $\xrightarrow{AB}$ is represented by $\langle b_1 - a_2, b_2 - a_2, \dots, b_n - a_n \rangle$ algebraicly.

(6) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$. Addition is commutative. (Consider the parallelogram diagram.)

(7) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$. Addition is associative. The parallelogram above becomes a parallelepiped:



(8) Why does the parallelogram rule correspond to the algebraic vector addition? Because the geometrically a parallelogram with vertices $(0,0),(a_1,a_2),(b_1,b_2)$ has the last vertice (a_1+b_1,a_2+b_2) given that (a_1,a_2) and (b_1,b_2) are not adjacent. This corresponds to the vector addition $\langle a_1,a_2\rangle+\langle b_1,b_2\rangle=\langle a_1+b_1,a_2=b_2\rangle$.

Dot Products

Taking **dot products** is a way to multiply vectors and get a scalar output.

(1) Algebraic representations: if $\vec{a}=\langle a_1,a_2,\dots,a_n\rangle$ and $\vec{b}=\langle b_1,b_2,\dots,b_n\rangle$, then

$$\vec{a}\cdot\vec{b}:=\sum_{i=1}^{n}a_ib_i=a_1b_1+a_2b_2+\cdots+a_nb_n.$$

(2) Basic properties: commutative, distributive, associative *with a scalar*: $(c\vec{a})\cdot b=c(\vec{a}\cdot\vec{b})=\vec{a}\cdot(c\vec{b})$; $\vec{0}\cdot\vec{a}=0$.

(3) Geometric representations: length and angle.

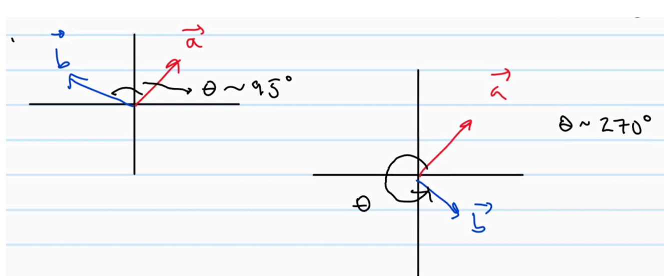
(4) If $\vec{a}=\langle a_1,a_2,\dots,a_n\rangle$, then $\|\vec{a}\|=\sqrt{\vec{a}\cdot\vec{a}}=\sqrt{\sum_{i=1}^{n}a_i^2}$.

6 Wed 8/26

What about angles, angles between vectors?

(1) In $\mathbb{R}^n$ where $n\geqslant 2$, any two nonzero vectors form an angle $\theta\in[0,2\pi]$. Same direction if $\theta=0$; pointing in opposite direcitons if $\theta=2\pi$. Ordering of vectors doesn't matter.

(2) In 2 dimensions, any two nonzero vectors **v** and **w** have an angle from **v** to **w**, with $\theta\in[0,2\pi]$:



(3) If we swap $\mathbf{v}$ and $\mathbf{w}$ then the angle θ gets replaced by $2\pi - \theta$.

(4) We will focus on angles between $[0, \pi]$ for now because we want to relate to dot products.

(5) The idea is based on *Law of Cosines*: for a triangle, $C^2 = A^2 + B^2 - 2AB\cos\theta$ where θ is the angle between sides A and B.

(6) Let $A := \|\vec{a}\|, B := \|\vec{b}\|$, and $C := \|\vec{a} - \vec{b}\|$. Then

$$\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\text{On the other hand, } \|\vec{a} - \vec{b}\|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \|\vec{a}\|^2 - 2\,\vec{a} \cdot \vec{b} + \|b\|^2$$

Therefore we have $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$.

(7) Very important: this gives up the geometric meaning of dot product in terms of lengths and angles.

(8) True in n dimensions, assuming $\theta \in [0, \pi]$.

(9) A version that I prefer:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}.$$

Problem 5

Find the angle between $\langle 4, 1, -2 \rangle$ and $\langle 3, 2, 0 \rangle$.

Solution

Since $\|\vec{a}\| = \sqrt{16 + 1 + 4} = \sqrt{21}$ and $\|\vec{b}\| = \sqrt{9 + 4 + 0} = \sqrt{13}$ snf $\vec{a} \cdot \vec{b} = 4 \cdot 3 + 1 \cdot 2 + 0 = 14$, we have

$$\cos\theta = \frac{14}{\sqrt{21} \cdot \sqrt{13}} \implies \theta = \cos^{-1}\left(\frac{14}{\sqrt{21}\sqrt{13}}\right) \approx 32.1°.$$

Problem 6

If we know $\|\vec{a}\| = 2$, $\|\vec{b}\| = 4$, and angle between them is $\frac{\pi}{3}$, what's $\vec{a}\vec{b}$?

Solution

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta = (2)(4)\cos\left(\frac{\pi}{3}\right) = 4.$$

Orthogonal vectors

The angle $\frac{\pi}{2}$ or 90° is very important. If the angle between $\vec{a}$ and $\vec{b}$ is a right angle, we say they are **perpendicular** or **orthogonal**. Write this as $\vec{a} \perp \vec{b}$. It is so special because two orthogonal vectors have dot product 0.

For two nonzero vectors $\vec{a}, \vec{b}$, we always have the following statement:

$$\theta = \frac{\pi}{2} \iff \vec{a} \cdot \vec{b} = 0$$

by the formula for $\cos\theta$ above in (9).

7 Fri 8/28 10.3 Projections, 10.4 Cross Products

Orthogonal projections

~ orthogonal projections of a vector $\mathbf{b}$ in $\mathbb{R}^n$ onto the line spanned by another vector $\mathbf{a}$: $\text{proj}_{\mathbf{a}}(\mathbf{b})$, **vector** projection of $\mathbf{b}$ onto the line spanned by $\mathbf{a}$. The "signed" length (relative to $\mathbf{a}$) is called the **scalar projection** of $\mathbf{b}$ onto line spanned by $\mathbf{a}$.

vector projection = (scalar projection) $\cdot(\mathbf{a}/\|\mathbf{a}\|)$.

Scalar projection > 0 if same direction, < 0 if opposite direction.

By the definition of $\cos\theta$ the length of the projected vector, or the scalar projection, is,

$$\|\text{proj}_{\mathbf{a}}(\mathbf{b})\| = \|\mathbf{b}\| \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}.$$

And therefore the vector projection is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} \cdot \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$$

Remark

If $\mathbf{a}$ is a unit vector $\hat{u}$ then the scalar projection of $\mathbf{b}$ simply becomes $\hat{u} \cdot \mathbf{b}$. Likewise, the vector projection becomes $(\hat{u} \cdot \mathbf{b})\hat{u}$.

Example 7.1

Compute the scalar and vector projections of $\mathbf{b} = \langle 1, 2, 3 \rangle$ onto $\mathbf{a} = \langle 4, 5, 6 \rangle$.

Solution

$\mathbf{a} \cdot \mathbf{b} = 32$ and $\|\mathbf{a}\|^2 = 77$. Then, the vector projection is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{32}{77} \langle 4, 5, 6 \rangle$$

and the scalar projection is

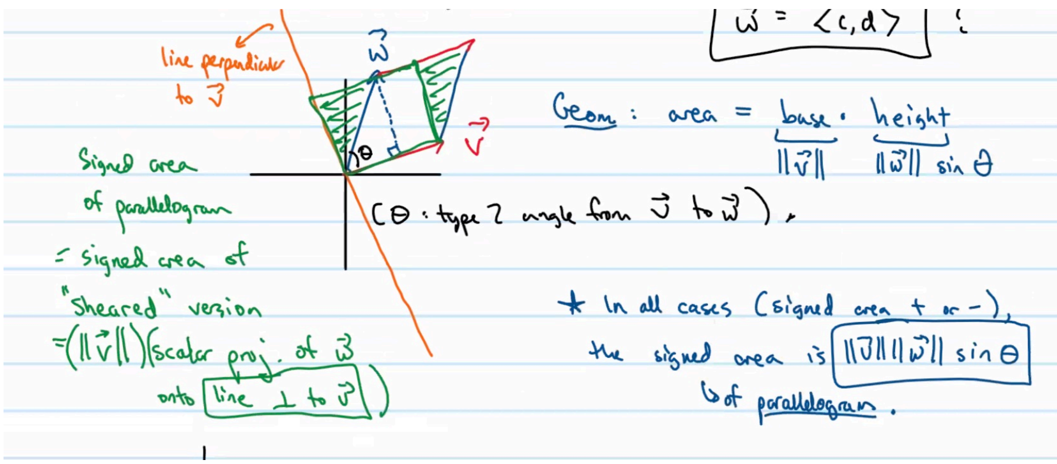
$$\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} = \frac{32}{\sqrt{77}}.$$

7.1 Cross products

First, a bit about determinants.

(1) in $\mathbb{R}^2$, given $\mathbf{a},\mathbf{b} \in \mathbb{R}^2$, consider the *signed* area of parallelogram spanned by $\mathbf{a},\mathbf{b}$. Define θ as the angle from $\mathbf{a}$ to $\mathbf{b}$. If $\theta \in [0,\pi]$, the area is nonnegative; if $\theta \in [\pi, 2\pi]$, the area is nonpositive.

(2) Suppose $\mathbf{v} = \langle a, b\rangle$ and $\mathbf{w} = \langle c, d\rangle$.What's the signed area of the parallelogram spanned by these two vectors?



(3) We don't want $\sin\theta$. Instead, consider the area of the parallelogram as the signed area of $\mathbf{v}$ and the vector projection of $\mathbf{w}$ onto the line orthogonal to $\mathbf{v}$ with intersection at origin.

(4) In $\mathbb{R}^2$, a 90° counterclockwise rotation of $\mathbf{v} = \langle a, b\rangle = \langle -b, a\rangle$. The ccw rotation matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(5) Therefore $\mathbf{v}$ rotated 90° ccw is $\mathbf{v}' = \langle -b, a\rangle$.

(6) Note that $\|\mathbf{w}\|\sin\theta = \|\mathbf{w}\|\cos(90^\circ - \theta)$ and $\|\mathbf{v}\| = \|\mathbf{v}'\|$. Therefore

$$\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = \|\mathbf{v}'\|\|\mathbf{w}\|\sin(90^\circ - \theta).$$

where the angle between $\mathbf{v}'$ and $\mathbf{w}$ is precisely $90^\circ - \theta$. Therefore this questions now becomes finding the dot product between these two vectors.

(7) The area is simply $\mathbf{v}'\cdot\mathbf{w} = \langle c, d\rangle \cdot \langle -b, a\rangle = ad - bc = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

8 Mon 8/31

Key properties of determinants:

(1) For a real number k, we have

$$\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and } \begin{vmatrix} a & kb \\ c & kd \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

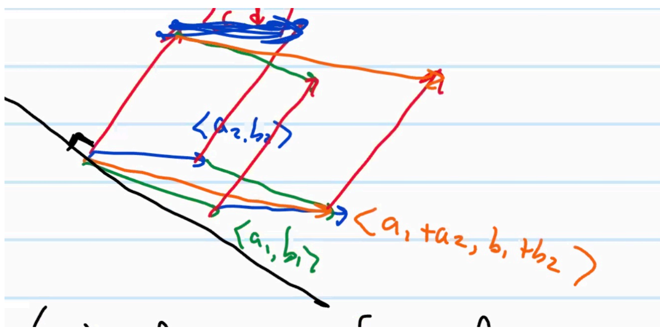
(2) Switching row flips the sign:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

(3) Addition of determinant is linear when **one** row is under addition:

$$\begin{vmatrix} a_1 + a_2 & b \\ c_1 + c_2 & d \end{vmatrix} = \begin{vmatrix} a_1 & b \\ c_1 & d \end{vmatrix} + \begin{vmatrix} a_2 & b \\ c_2 & d \end{vmatrix}$$

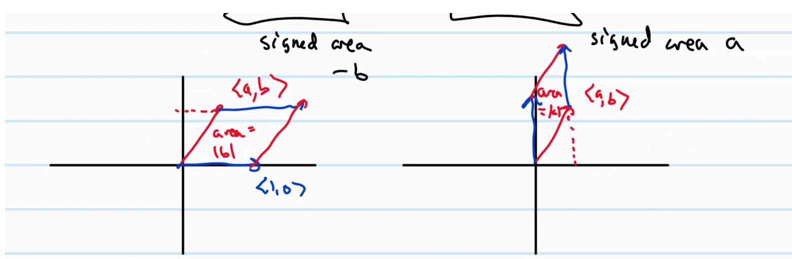
Geometric interpretation:



We can "shear" all parallelograms to become rectangles with base along this perpendicular line. Projection of sums is sum of projections. Hence it holds.

if we assume $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$:= signed area of parallelogram, we know that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = c \begin{vmatrix} a & b \\ 1 & 0 \end{vmatrix} + d \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad - bc$$



(4) The volume of parallelepiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ has positive signed volume if $\mathbf{c}$ is on the same side of plane by $\mathbf{a}$ and $\mathbf{b}$ by the right hand rule. Negative otherwise.

(5) † Prof. Manion prefers to write vectors as column vectors. (Convention in Linear Algebra?) From now on I will adapt his way. Anyway the volume is the same anyway: the determinant is the same for transpose:

$$V = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(6) The signed volume = area of parallelogram spanned by $\mathbf{a}, \mathbf{b}$ times the scalar projection of $\mathbf{c}$ onto **the line perpendicular to** the plane spanned by $\mathbf{a}, \mathbf{b}$.

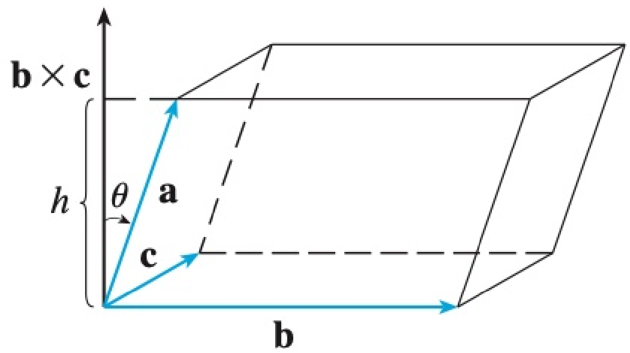
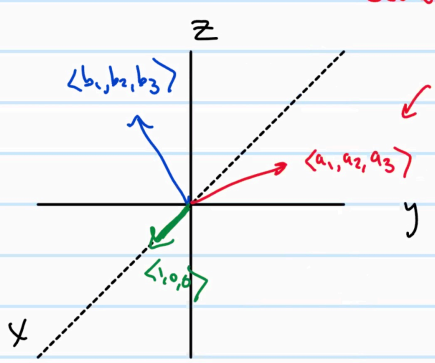


Diagram from book; different names for vectors

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c_1 \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} + c_2 \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 0 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 1 \end{vmatrix}$$

(7) For the first one with c_1, the signed volume of the sub-parallelepiped is the same as a parallelepiped with base $\langle 0, a_2, a_3 \rangle$ and $\langle 0, b_2, b_3 \rangle$ on the yz-plane ignoring the x-component and height 1 from $\langle 1, 0, 0 \rangle$.

Geometrically:



Signed vol of parallelepiped spaned by these 3 vectors: unchanged if we replace

$\langle a_1, a_2, a_3 \rangle \to \langle 0, a_2, a_3 \rangle$

and

$\langle b_1, b_2, b_3 \rangle \to \langle 0, b_2, b_3 \rangle$.

We've reduced to computing signed vol of parallepiped spanned by $\langle 0, a_2, a_3 \rangle$, $\langle 0, b_2, b_3 \rangle$, $\langle 1, 0, 0 \rangle$.

(8) Likewise for the other 2 sub-parallelepipeds. Therefore

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

(9) Signs: even row + column = +, odd = −.

9 Wed 9/2

The 3D determinant is a dot product in the following form: specifically, it's

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \langle c_1, c_2, c_3 \rangle \cdot \left\langle \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right\rangle = \langle c_1, c_2, c_3 \rangle \cdot (\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle)$$

On the other hand, we can build a vector geometrically from $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ such that the dot product with this new vector is also equal to signed volume of the parallelepiped. Idea: volume = (area of base) · (signed height above/below the base). Refer to the diagram on the previous page, copied from textbook.

Order between the base and the height doesn't matter:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Anti-commutativity: $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$. Thus $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$.

Example 9.1

Compute $\langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle$ and check orthogonality.

Solution

$$\langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle = \begin{vmatrix} \mathbf{j} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \left\langle \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}, -\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \right\rangle = \langle -3, 6, -3 \rangle$$

and indeed $\langle 1, 2, 3 \rangle \cdot \langle -3, 6, -3 \rangle = 0 = \langle 4, 5, 6 \rangle \cdot \langle -3, 6, -3 \rangle$.

Problem 7

What's the area of the parallelogram spanned by $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 6 \rangle$?

Solution

It's simply $\|\langle 1,2,3\rangle \times \langle 4,5,6\rangle\| = \sqrt{(-3)^2+6^2+(-3)^2} = \sqrt{54} = 3\sqrt{6}$.

Problem 8

What's the volume of the parallelepiped spanned by $\langle 1,2,3\rangle, \langle 4,5,6\rangle, \langle 7,8,9\rangle$?

Solution

It's simply $\langle 7,8,9\rangle \cdot (\langle 1,2,3\rangle \times \langle 4,5,6\rangle) = \langle 7,8,9\rangle \cdot \langle -3,6,-3\rangle = 0$[!] Therefore these three vectors are coplanar, i.e., living in a common plane in $\mathbb{R}^3$.

Remark

The scalar triple product or the 3D determinant (signed volumes) can be used to detect whether three vectors are coplanar. Similarly, the cross product or the area of parallelogram spanned by two vectors can detect whether these two vectors are colinear.

10 Fri 9/4 10.4 Wrap-up & 10.5 Lines and Planes

10.4 Wrap-up

Last time: cross product of vectors in $\mathbb{R}^3$; algebraic and geometric meaning of scalar triple product.

Remark

$\|\mathbf{a} \times \mathbf{b}\|$ is the area of the 2D parallelogram in 3D space spanned by **a** and **b**. In 2D, $\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$ is also a formula that gives the area of a parallelogram.

Remark

$\times$ is not associative!

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \text{ in general. The RHS is called the } \textbf{vector triple product.}$$

There's a formula, **Lagrange expansion / triple product expression**, for vector triple product:

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \text{ (scalar times vector)}\\ &= \mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \text{ (no cross products anymore!).}\end{aligned}$$

Why? $\mathbf{b}\times\mathbf{c}$ is a vector along the line L that's perpendicular to the plane spanned by $\mathbf{b}$ and $\mathbf{c}$ (through the origin). Therefore $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is perpendicular to L, i.e., it's *in* that same plane spanned by $\mathbf{b}$ and $\mathbf{c}$.

Of course we also know that this vector triple product is also perpendicular to $\mathbf{a}$. It's in the plane through the origin perpendicular to $\mathbf{a}$.

Case 1: If the plane perpendicular to L and to $\mathbf{a}$ are the same, then $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}=0$ since $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$ are colinear. In this case $\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})=0\mathbf{b}-0\mathbf{c}=0$.

Case 2: If the plane perpendicular to L and to $\mathbf{a}$ are not the same, then they intersect in a line L', and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ lives in this line L'. On the other hand, $\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ also lives in the line because 1) it's perpendicular to L as it lives in the plane of $\mathbf{b}$ and $\mathbf{c}$ and 2) it's perpendicular to $\mathbf{a}$ because

$$\mathbf{a}\cdot(\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}))=(\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c})-(\mathbf{a}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})=0$$

Now we know that they lie on the same line, and *it's left as an exercise for readers to verify that* the vector triple product and its Lagrange expansion are exactly the same.

10.5 Lines and Planes

Recall that when we are given a shape, we can try to

(1) describe it as the graph of some function F, e.g., $f(x)=mx+b$

(2) describe it as a level set of some function F, e.g., $L_0(F(x,y):=ax+by)$

(3) (this §) describe it as the image of some function F

Problem 9

Let L be the line in $\mathbb{R}^2$ through the point (x_0,y_0) and perpendicular to the vector $\langle a,b\rangle$. How can we write L as a level set?

Solution

First, in a degenerate case that $(x_0,y_0)=(0,0)$, then

$$L=\{(x,y)\mid\langle x,y\rangle\cdot\langle a,b\rangle\}=0\implies\{(x,y)\mid ax+by=0\}\implies L_0(F(x,y):=ax+by=0)$$

Now if (x_0, y_0) is not at the origin, then we simply replace $ax + by = 0$ by $a(x - x_0) + b(y - y_0) = 0$ since now the two orthogonal vectors become $\langle a, b\rangle$ and $\langle x - x_0, y - y_0\rangle$. Then this line can be described as

$$L_0(F(x,y) \coloneqq a(x - x_0) + b(y - y_0)) \text{ or } L_{(ax_0+by_0)}(F(x,y) \coloneqq ax + by)$$

Problem 10

Let L be the line $\mathbb{R}^2$ through the point (x_0, y_0) and in the direction of $\langle a, b\rangle$. Write L as a level set.

Solution

"In direction of $\langle a, b\rangle$" is the same as "perpendicular to $\langle -b, a\rangle$" (in 2D). Then everything just follows from the problem above:

$$\left\{(x,y) \mid \frac{x - x_0}{a} = \frac{y - y_0}{b}\right\}, \text{ a symmetric equation of the line in a level set perspective.}$$

Remark

We started with $\mathbb{R}^2$. If we impose impose one linear constraint, this cuts the dimension down by 1. From $\mathbb{R}^2$ to a 1D line, for example. Or from $\mathbb{R}^3$ to a 2D plane. If we impose two constraints, then we reduce $\mathbb{R}^3$ to a 1D line.

11 Wed 9/9 More on level sets and planes

In $\mathbb{R}^3$, any equation of the form $ax + by + cz = d$ defines a lane in $\mathbb{R}^3$. More specifically, this plane has normal vector $\langle a, b, c\rangle$.

Problem 11

Write P as a level set where P is the plane in $\mathbb{R}^3$ through the point (x_0, y_0, z_0) and normal to the vector $\langle a, b, c\rangle$.

Solution

Quite similar to the one example in 2D...If $(x_0, y_0, z_0) = (0, 0, 0)$, then P is the level set of $F(x, y, z) = ax + by + ez$ at level zero $(\langle x, y, z\rangle \cdot \langle a, b, c\rangle = 0$. If (x_0, y_0, z_0) is not the origin, then we can shift the origin and obtain $\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0 \implies ax + by + cz = ax_0 + by_0 + cz_0$. The RHS is a constant and will be the "level" of interest of the level set. It can also be written as the scalar equation

$$a(x - x_0) + b(x - x_0) + c(x - x_0) = 0$$

or, alternatively, in vector form: $F(\mathbf{r}) = \mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0})$ where $\mathbf{n} = \mathbf{r} \langle a, b, c \rangle, \langle x, y, z \rangle$ and $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$.

Problem 12

Write P as a level set where P is the plane in $\mathbb{R}^3$ through point (x_0, y_0, z_0) and containing the directions $\mathbf{v}, \mathbf{w}$, two vectors in $\mathbb{R}^3$.

Solution

If the plane contains $\mathbf{v}, \mathbf{w}$, then $\mathbf{v} \times \mathbf{w}$ is normal to the plane. Then we have a normal vector and we have a specific point (x_0, y_0, z_0) on the plane. Ezpz.

Recall that if we were to write lines in $\mathbb{R}^3$ as level sets, we have to impose 2 constraints to reduce $\mathbb{R}^3$ to $\mathbb{R}$.

Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ with $F(x, y, z) = (a_1 x + b_1 y + c_1 z + d_1, a_2 x + b_2 y + c_2 z + d_2)$. Level sets of the components are both planes respectively. Therefore if the two planes are not parallel, they intersect, and the intersection is always a line in $\mathbb{R}^3$. Therefore the level sets of F are lines in $\mathbb{R}^3$.

Problem 13

Write L as a level set where L is the line in $\mathbb{R}^3$ through the point (x_0, y_0, z_0) and perpendicular to both $\langle a_1, b_1, c_1 \rangle$ and $\langle a_2, b_2, c_2 \rangle$.

Solution

As stated before, we should have the system of equations

$$a_1(x - x_0) + b_1(y - y_0) + c_1(z - z_0) = 0 \quad i.e., \langle a_1, b_1, c_1 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$
$$a_2(x - x_0) + b_2(y - y_0) + c_2(z - z_0) = 0 \quad i.e., \langle a_2, b_2, c_2 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Now we can encode this as a level set of $F : \mathbb{R}^3 \to \mathbb{R}^2$. Then line L can be expressed as a level set of

$$F(x, y, z) = (a_1(x - x_0) + b_1(y - y_0) + c_1(z - z_0), a_2(x - x_0) + b_2(y - y_0) + c_2(z - z_0)) \text{ at level } (0, 0).$$

Problem 14

Write L as a level set where L is the line in $\mathbb{R}^3$ through (x_0, y_0, z_0) in the direction of $\langle a, b, c\rangle$.

Solution

A vector is in the direction of $\langle a, b, c\rangle$ if and only if it's perpendicular to both $\langle -b, a, 0\rangle$ and $\langle 0, -c, b\rangle$. So one possible system of equations can be obtained from the previous problem.

$$\begin{aligned} -b(x-x_0) \quad +a(y-y_0) \qquad\qquad &= 0 \\ -c(y-y_0)+b(z-z_0) &= 0 \end{aligned}$$

Then $F(x, y, z) = (\sim, \sim)$ at level $(0, 0)$ is a level set expression of the line. Then this becomes

$$\begin{cases} b(x-x_0) = a(y-y_0) \\ c(y-y_0) = b(z-z_0) \end{cases} \implies \text{if } a, b, c \text{ all nonzero: } \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}, \text{ symmetric equations.}$$

Parametrization

Now we add to our list of ways to either visualize a function or *understand a shape in terms of a function.* Again recall that so far we know to

(1) view the shape as a graph of some function

(2) view shape as a level set of some function (which we just did)

(3) *view shape as the image of some function* — **parametrization.**

If $F : \mathbb{R}^n \to \mathbb{R}^m$: we can

(1) visualize the graph if and only if $n + m \leqslant 3$

(2) visualize the level sets if and only if $n \leqslant 3$

(3) visualize image if and only if $m \leqslant 3$

Problem 15

Write L as an image, i.e., parametrize L, where L is the line in $\mathbb{R}^2$ through (x_0, y_0) in the direction of $\mathbf{v} = \langle a, b\rangle$.

Solution

Line L is everything of form $\mathbf{r_0} + t\mathbf{v}$ where $\mathbf{r_0} = \langle x_0, y_0 \rangle$ and $\mathbf{v} = \langle a, b \rangle$. The line is the image of $F : \mathbb{R} \to \mathbb{R}^2$ with $\mathbf{F}(t) = \mathbf{r_0} + t\mathbf{v} = \langle x_0 + at, y_0 + bt \rangle$, and the parametric equations can be written as

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}$$

12 Fri 9/11 10.5 Wrap Up and a bit of 10.6

Last time:

Q:how to write L as an image where L is the line in $\mathbb{R}^2$ through $\mathbf{r_0} = \langle x_0, y_0 \rangle$ and in direction of $\mathbf{v} = \langle a, b \rangle$?

A: L is the image of $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ where $\mathbf{F}(t) = \mathbf{r_0} + t\mathbf{r}$.

Equivalently, the components of $\mathbf{F}$ can be written as $x_0 + at$ and $y_0 + bt$.

Q: same question but for line L in $\mathbb{R}^3$ through $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$ and in the direction of $\mathbf{v} = \langle a, b, c \rangle$:

A: $\mathbf{F}(t) = \mathbf{r_0} + t\mathbf{v}$ with components $x_0 + at, y_0 + bt, z_0 + ct...$ or the line described by the parametric equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

Problem 16

How about writing P as the image of a function, where P is the plane in $\mathbb{R}^3$ through the point (x_0, y_0, z_0) (so $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$) and containing the directions of $\mathbf{v}, \mathbf{w}$ (that being said, $\mathbf{v}, \mathbf{w}$ are not colinear).

Solution

P is the image of $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ where

$$\mathbf{F}(s, t) = \mathbf{r_0} + s\mathbf{v} + t\mathbf{w}.$$

Above is basically the core of 10.5.

Problem 17

Consider line L in $\mathbb{R}^3$ as an image. How about writing the equations of a line segment S between $\mathbf{r_0}, \mathbf{r_1}$ as an image?

Solution

S is part of the line L. Immediately we know two things:

(1) L contains $\mathbf{r_0}$

(2) L is in the direction of $\mathbf{r_1} - \mathbf{r_0}$.

Then the line L is defined by

$$\mathbf{F}(t) = \mathbf{r_0} + t(\mathbf{r_1} - \mathbf{r_0}) = (1-t)\mathbf{r_0} + t\mathbf{r_1}$$

and the segment S is the convex combination of these two combinations:

$$\mathbf{F}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1} \text{ where } t \in [0,1]$$

Thus, the segment S is the image of $\mathbf{G} : [0,1] \to \mathbb{R}^3$ defined by $\mathbf{G}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1}$. Note that $\mathbf{G}$ and $\mathbf{F}$ are different because they do not have the same domain. More specifically, $\mathbf{G}$ is a restriction of $\mathbf{F}$.

Problem 18

Given 3 points in $\mathbb{R}^3$, P_1, P_2, P_3, write the plane through them as a level set.

Solution

Define $\mathbf{v} = P_2 - P_1$ and $\mathbf{w} = P_3 - P_1$. Clearly $\mathbf{v}$ and $\mathbf{w}$ lie in the plane. Therefore one normal vector of the plane can be obtained by taking $\mathbf{v} \times \mathbf{w}$. Thus the plane can be written as $F(x,y,z) = \left\langle \langle x,y,z\rangle - \vec{P}_1 \right\rangle \cdot (\mathbf{v} \times \mathbf{w})$ at level 0.

Problem 19

What's the angle between two planes?

Solution

The angle between the planes is the same as the angle between the normal vectors. Once we get the normal vectors $\mathbf{v}, \mathbf{w}$ then

$$\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right).$$

Problem 20

Find the distance between a point $P = (x_1, y_1, z_1)$ in $\mathbb{R}^3$ and the plane $ax + by + cz + d = 0$ in $\mathbb{R}^3$.

Solution

First pick $P_0 = \langle x_0, y_0, z_0 \rangle$ in the plane. Then $ax_0 + by_0 + cz_0 + d = 0$. Then the distance between P_1 to the plane is the same as $d = \|\overrightarrow{P_1 P_0}\| \cos\theta$ where θ = the angle between $\overrightarrow{P_1 P_0}$ and the normal vector $\langle a, b, c \rangle$. Then,

$$d = \|P_1 P_0\| \left(\frac{\overrightarrow{P_1 P_2} \cdot \langle a, b, c \rangle}{\|\overrightarrow{P_1 P_0}\| \| \langle a, b, c \rangle \|}\right) = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Briefly: 10.6 Cylinders and Quadric Surfaces

Why are lines and planes special? Because they are level sets (at level 0) of special "linear" (affine) functions – more specifically, polynomial functions of degree $\leqslant 1$.

(1) Lines are level sets of single-variable polynomials with degree 1.

(2) A quick definition by example: $f(x, y) = x^3 y + x^3 + y^2$ is a multi-variable polynomial of degree 4 as all variables count equally towards the degree. This example is a quartic polynomial in 2 variables.

(3) Planes are level sets of multi-variable polynomials with degree 1.

(4) Quadric surfaces replaces with the statement "deg $\leqslant 1$" by "deg $\leqslant 2$": they are the zero sets in $\mathbb{R}^3$ of polynomial functions $f : \mathbb{R}^3 \to \mathbb{R}$ of degree $\leqslant 2$.

13 Mon 9/14 10.6/7: Quadrics/Cylinders, Functions $\mathbf{F} : \mathbb{R} \to \mathbb{R}^m$

Last time: polynomials in multiple variables

Conic sections: zero sets (or more general level sets) of degree 2 polynomial in 2 variables: ellipses, hyperbolas, parabolas, degenerate cases.

Quadric surfaces: the 3D equivalent of conic sections: the zero sets (or more general level sets) of degree 2 polynomials in 3 variables:

$$F(x,y,z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J$$

After changes of variables, this has the form

$$F(x,y,z) = Ax_0^2 + By_0^2 + Cz_0^2 + J \text{ or } F(x,y,z) = Ax_1^2 + By_1^2 + Iz$$

The first one has $\varnothing$ as zero set unless $J \leqslant 0$, and the second has $\varnothing$ as zero set unless $I \leqslant 0$.

Quadratic (polynomials) means polynomials of degree 2, whereas quadric is defined above. Quartic (polynomials) means polynomials of degree 4.

Three types corresponding to the first type of equation: eppilsoids, hyperboloid of one sheet (two among ABC are positive), or hyperboloid of two sheets (one positive). A cone is a special hyperboloid.

Two types corresponding to the second type of equation: elliptic paraboloid, hyperbolic paraboloid,

Missing one variable: e.g. $F(x,y,z) = x^2 + z^2 - 1$, the image becomes a cylinder. If we fix that missing variable we get a ellipse (or a circle in a special case). **Rulings** are the set of points with two variable fixed and the third missing "free variable" freely moving.

Same idea applies even for level sets of non-polynomial functions: for example consider $F(x,y,z) = z - \sin(y)$. The graph looks like sinusoidal waves on the yz-plane that gets extruded along x-axis.

Calculus of functions $\mathbf{F}:\mathbb{R}\to\mathbb{R}^n$

Main point: $\mathbf{F}:\mathbb{R}\to\mathbb{R}^m$ has components

$$\mathbf{F}(t) = \langle F_1(t), \dots, F_n(t)\rangle$$

where each component is defined by a single-variable function. Then

$$\text{Calculus for } \mathbf{F} \iff \text{Calculus for each } F_i : \text{derivatives, integrals, etc.}$$

For now, we will not focus too much on limits or proofs.

Briefly: $\epsilon - \delta$ definition of continuity.

Definition 1: Limits

For $\mathbf{F}:\mathbb{R}\to\mathbb{R}^m$, we say

$$\lim_{t\to t_0} \mathbf{F}(t_0) = \mathbf{L}$$

if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$t \in (t_0 - \delta, t_0 + \delta) \implies \|\mathbf{F}(t) - \mathbf{L}\| < \epsilon.$$

If $\mathbf{L} = \langle L_1, \ldots, L_m \rangle$ then $\lim_{t \to t_0} F_i(t) = L_i$.

Definition 2

If $f : \mathbb{R} \to \mathbb{R}$, we say f is **continuous at** t_0 if $\lim_{t \to t_0} f(t)$ exists and is equal to t_0. f is continuous if it's continuous for all t_0 in its domain.

Definition 3

$\mathbf{F} : \mathbb{R} \to \mathbb{R}^m$ is continuous at t_0 if $\lim_{t \to t_0} \mathbf{F}(t)$ exists and equals $\mathbf{F}(t_0)$. $\mathbf{F}(x)$ is continuous if it's continuous for all t_0 in its domain.

Remark

Another way to visualize functions is to imagine a point moving through the m-dimensional space, and this encodes *all* the data of the particle such as its speed/derivative.

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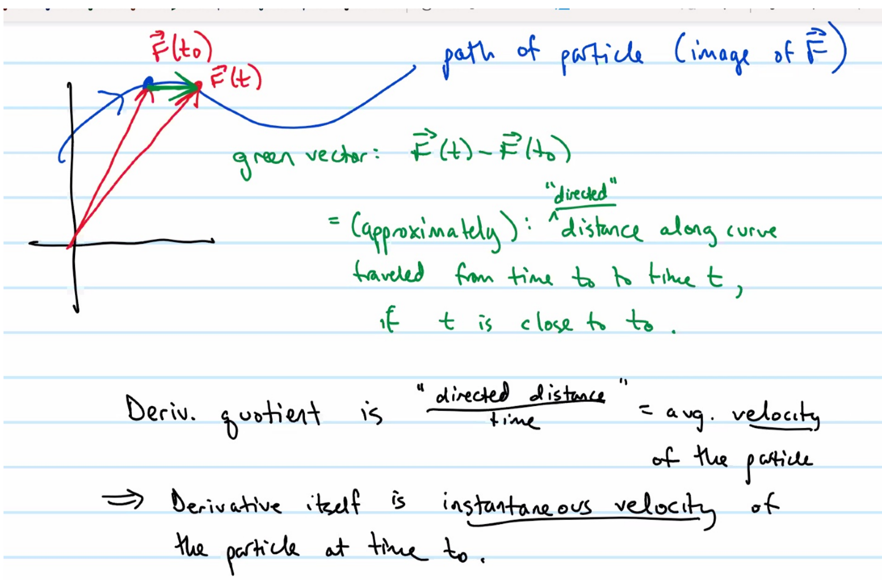
Last time: $\mathbf{F}(t)$ is continuous if and only if there are no "jumps" in the "motion" of the particle described by $\mathbf{F}$.

Definition 4

We say $\mathbf{F}: \mathbb{R} \to \mathbb{R}^m$ is **differentiable at** t_0 if $\lim_{t \to t_0} \frac{\mathbf{F}(t) - \mathbf{F}(t_0)}{(t - t_0)}$ exists.

Remark: Very long remark

The **derivative** is highly analogous to that of a single-variable function. See screenshot below.



So are the definition of higher order derivatives, if they exist.

In terms of components, if $\mathbf{F}(t) = \langle F_1(t), F_2(t), \ldots, F_m(t) \rangle$, then $\mathbf{F}'(t_0)$ exists if and only if all $F_i(t_0)$'s all exist. In this case we have the derivative

$$\mathbf{F}'(t_0) = \langle F_1'(t_0), \ldots, F_m'(t_0) \rangle .$$

In terms of the "movie" intuition, it is the velocity vector of the particle at time t_0.

In terms of image: if we know the curve but *not* the parametrization, we can find the tangent vectors and tell the direction, but *not* how fast the particle is traveling. Consider $y = x$, a curve, and two parametrizations:

$$\begin{cases} x = t \\ y = t \end{cases} \quad \text{and} \quad \begin{cases} x = t^2 \\ y = t^2 \end{cases}$$

Both of them works, and clearly the particle have different speed.

If we pick an **orientation**, the direction of travel for curve, then we have a unique **unit tangent vector** in the direction of that travel.

Now suppose I *do* have $\mathbf{F} : \mathbb{R} \to \mathbb{R}^m$ parametrizing this curve, such that $\mathbf{F}$'s direction of travel (assuming this makes sense) is given. We cannot interpret $\mathbf{F}'(t_0)$ in terms of image, only in terms of the movie.

However, $\mathbf{F}'(t_0)/\|\mathbf{F}'(t_0)\|$ *does* make sense in terms of the oriented image: the unit tangent vector in the direction of travel.

Some rules:

(1) $(\mathbf{F} + \mathbf{G})' = \mathbf{F}' + \mathbf{G}'$.

(2) $(c\mathbf{F})' = c(\mathbf{F}')$.

(3) **Leibniz Product Rule** for single variable: $(fg)' = (f')g + f(g')$.

Some versions for multi variable:

(1) Scalar multiplication: $f : \mathbb{R} \to \mathbb{R}$, $\mathbf{G} : \mathbb{R} \to \mathbb{R}^m$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(t)\mathbf{G}(t)) = \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{G}(t) + f(t)\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}}(t).$$

(2) Dot product: $\mathbf{F}, \mathbf{G} : \mathbb{R} \to \mathbb{R}^m$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}(t) \cdot \mathbf{G}(t)) = (\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}(t)) \cdot \mathbf{G}(t) + \mathbf{F}(t) \cdot (\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t}(t))$$

(3) Cross product: $\mathbf{F}, \mathbf{G} : \mathbb{R} \to \mathbb{R}^3$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}(t) \times \mathbf{G}(t)) = (\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}(t)) \times \mathbf{G}(t) + \mathbf{F}(t) \times (\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t}(t))$$

Finally, the chain rule: $\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{\mathbf{G}} \mathbb{R}^m$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{G}(f(t)) = (\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t}f(t))f'(t)$$

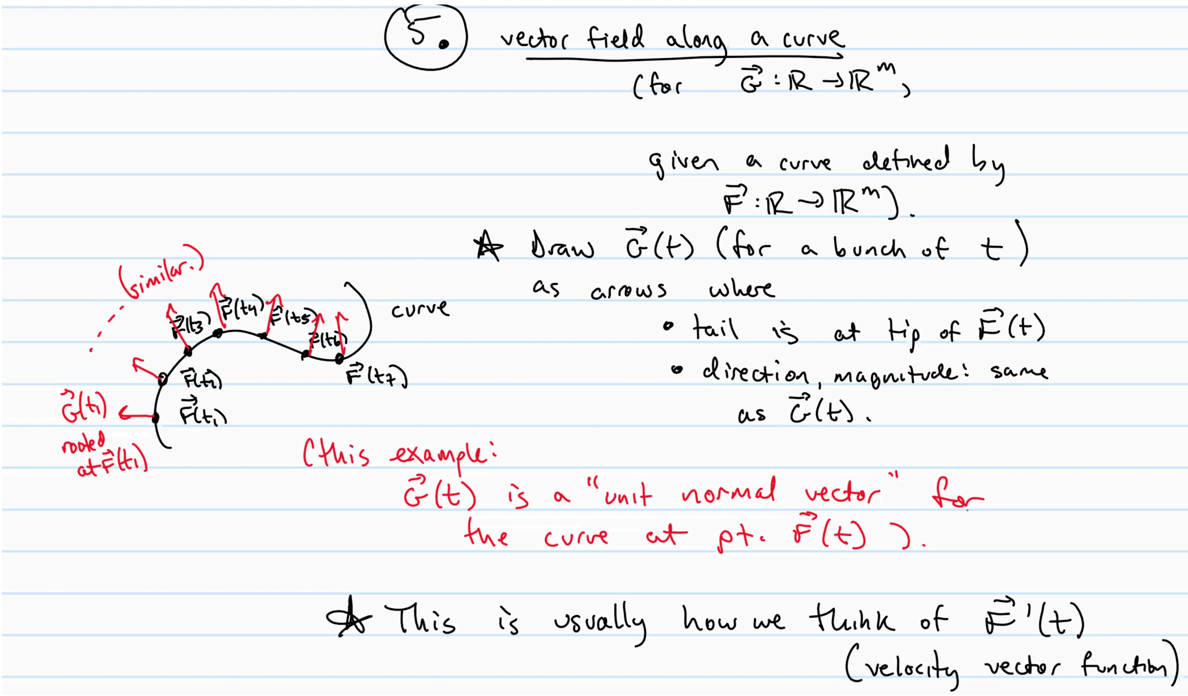
Example 14.1

Let C be the curve in $\mathbb{R}^3$ given as the image of $\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$ defined by $\mathbf{F}(t) = (t, t^2, t^3)$. Let L be the tangent line to C at the point $P = (1, 1, 1)$. Write L as an image.

Solution

Note that $P = (1,1,1)$ correspond to $t = 1$. The line would simply be of form $\mathbf{P} + t\mathbf{F'}(1) = \langle 1,1,1\rangle + t\langle 1,2,3\rangle$.

15 Fri 9/18



For integrals: definite integrals can be defined as "limit" over Riemann sums. For us, we will just take it for granted to integrate each component of $\mathbf{F}(t)$:

Example 15.1

Define $\mathbf{F}(t) = \langle \cos t, \sin(3t) \rangle$. Then

$$\int_0^{\pi/3} \mathbf{F}(t)dt = \left\langle \int_0^{\pi/3} \cos t dt, \int_0^{\pi/3} \sin(3t)dt \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{2}{3} \right\rangle$$

More generally, we view $\mathbf{F}(t)$ as the derivative $\mathbf{G}'(t)$ where $\mathbf{G}(t) \in \mathbb{R}^m$ is thought of a movie of a point in $\mathbb{R}^m$:

$$\int_{t_1}^{t_2} \mathbf{F}(t)dt = \mathbf{G}(t)\Big|_{t=t_1}^{t=t_2} = \langle G_1(t_2) - G_1(t_1), G_2(t_2) - G_2(t_1), \ldots, G_m(t_2) - G_m(t_1) \rangle = \mathbf{G}(t_2) - \mathbf{G}(t_1)$$

This is a displacement vector, a "straight line" between the two endpoints.

For indefinite integrals, each component function has a constant C_i, and they together become the constant vector $\mathbf{C} = \langle C_1, C_2, \ldots, C_m \rangle$.

How about $\int_{t_1}^{t_2} \|\mathbf{F}'(t)\| dt$? This is more dependent on the curve: how does $\mathbf{F}$ get from $\mathbf{F}(t_1)$ to $\mathbf{F}(t_2)$?

In fact, this is the **arc length** from $\mathbf{F}(t_1)$ to $\mathbf{F}(t_2)$. Note that $\int_{t_1}^{t_2} \|\mathbf{F}'(t)\| dt \geqslant \int_{t_1}^{t_2} \mathbf{F}'(t) dt$ since 1)norms are positive semidefinite and 2) the shortest distance between two points is the straight line.

Example 15.2

An example impossible to solve: define C, an ellipse, as

$$\mathbf{F}(t) = \langle a\cos t, b\sin t\rangle \text{ with } a \neq b.$$

Compute (yes, impossible) the arc length between $t \in [0, \frac{\pi}{2}]$, one fourth the perimeter of the ellipse with semiaxes a and b.

Solution

Since $\mathbf{F}'(t) = \langle -a\sin t, b\cos t\rangle$, we have

$$\|\mathbf{F}'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

and the quarter arc length is

$$\int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t} dt,$$

an elliptic integral that cannot be solved by elementary math.

We will skip **reparametrization by arc length** for now. It's the parametrization of a parametrization to make the "speed" vector 1 everywhere.

For HW: a limit in $\mathbb{R}^2$:

Problem 21

Let $F(x,y) = (xy)/(x^2+y^4)$. It is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Does

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^4}$$

exist?

Solution

Consider different lines of approach to $(0,0)$. If limits obtained by two different liens of approach is different, then the limit doesn't exist.

In this example, we pick the following:

$$\lim_{(x,0)\to(0,0)} \frac{x \cdot 0}{x^2 + 0^4} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$$

and

$$\lim_{(x,x)\to(0,0)} \frac{x \cdot x}{x^2 + x^4} = \lim_{(x,x)\to(0,0)} \frac{1}{1 + x^2} = 1.$$

Therefore the limit DNE.

To prove that the limit exist, more techniques need to be used.