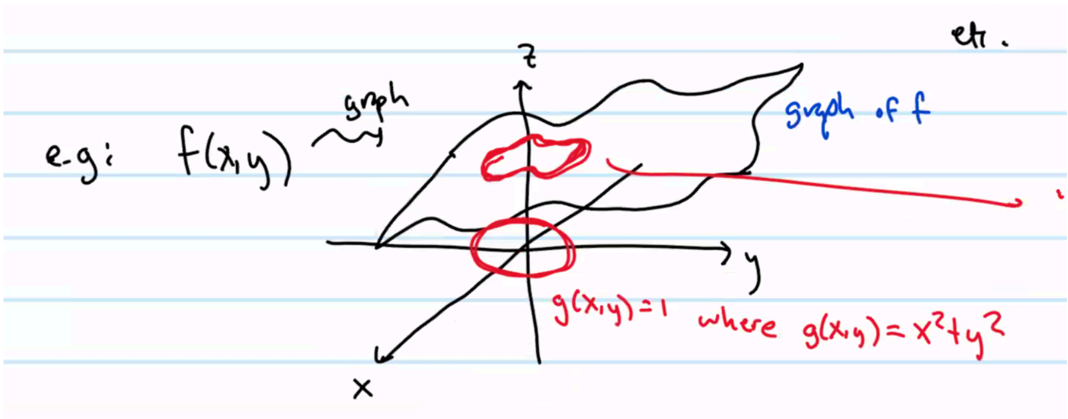
1 Mon 10/12 11.8

11.8 Lagrange Multipliers

General idea: maximize / minimize a scalar-valued function $f(x_1, x_2, \ldots, x_n)$ where $(x_1, \ldots, x_n)$ are subject to a constraint $g(x_1, x_2, \ldots, x_n) = k$ for some constant k. Furthermore, we can impose multiple constraints $g_1, g_2, \ldots,$ but then we have to solve more stuff of course.



Finding min/max for $f(x, y)$ restricted to S^1: the wiggling curve.

One way: analyze the global min and max of $f(x, \sqrt{1 - x^2})$ by first analyzing the interior and then the boundary, as what we've done previously. However most of the times this can be very computation-intensive.

The other way, the standard Calc III method: **Lagrange multipliers**.

Theorem 1

If (x, y) is a local min/max for f on all of $\mathbb{R}^2$ ($f : \mathbb{R}^2 \to \mathbb{R}$, then the gradient vanishes at such point, i.e., $\nabla f(x, y) = 0$.

Similarly, if (x, y) is a local min/max for f restricted to $g(x, y) = k$, a similar logic applies: though $\nabla f(x, y)$ may not be 0, it is "zero modulo $\nabla g(x, y)$". Namely, there exists $\lambda \in \mathbb{R}$ satisfying

$$\nabla f(x, y) = \lambda \nabla g(x, y).$$

Here the λ is called the **Lagrange Multiplier.**

Remark

But why? If $f(x,y)$ is a min/max for f on all of $\mathbb{R}^2$, then the directional derivative $D_{\mathbf{u}}f(x,y) = 0$ for all unit vectors $\mathbf{u}$. Similarly, if (x,y) is a min/max for f restricted to g, then

$D_{\mathbf{u}}f(x,y) = 0$ for all $\mathbf{u}$ tangent to the level set/curve $g(x,y) = k$, i.e., $\perp \nabla g(x,y)$ (recall dot product).

Therefore $\nabla f(x,y)$ is perpendicular to all unit vectors $\mathbf{u}$ that are perpendicular to $\nabla g(f,y)$. Hence $\nabla f(x,y)$ and $\nabla g(x,y)$ are parallel!

Corollary 2

Suppose $f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$ is now imposed with two constraints $g_1(x,y,z) = k_1$, $g_2(x,y,z) = k_2$. Then the newly restricted f attains min/max where

$$\nabla f(x,y,z) = \lambda_1 g_1(x,y,z) + \lambda_2 g_2(x,y,z) \text{ for some } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Remark

Think of this as a generalization of the previous example. Here $\nabla f(x,y,z)$ is perpendicular to all the unit vectors $\mathbf{u}$ that are perpendicular to both $\nabla g_1(f,y,z)$ and $\nabla g_2(x,y,z)$. Hence it is parallel to the span of these two gradient vectors.

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Consider $f : \mathbb{R}^2 \to \mathbb{R}$ whose graph looks like a surface. The column below the surface bounded by $D \subset \mathbb{R}^2$ is given by

$$\iint_D f(x,y)\,\mathrm{d}A.$$

If $D = [a,b] \times [c,d]$, then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

by *Fubini's Theorem*, which we will get to later on.

Example 2.1

Compute
$$\iint_{[1,4]\times[0,2]} (6x^2y - 2x)\ \mathrm{d}A$$
as an iterated integral.

Solution

$$\begin{aligned}\iint_{[1,4]\times[0,2]} 6x^2y - 2x\ \mathrm{d}A &= \int_1^4 \int_0^2 (6x^2y - 2x)\ \mathrm{d}y\ \mathrm{d}x \\ &= \int_1^4 \left[3x^2y^2 - 2xy\right]_{y=0}^{y=2}\ \mathrm{d}x \\ &= \int_1^4 12x^2 - 4x\ \mathrm{d}x \\ &= \left[4x^3 - 2x^2\right]_{x=1}^{x=4} \\ &= 222.\end{aligned}$$

Triple integral case:
$$\iiint_{[a,b]\times[c,d]\times[e,f]} f(x,y,z)\ \mathrm{d}V = \int_a^b \int_c^d \int_e^f f(x,y,z)\ \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}x = \dots \text{ by Fubini.}$$

Likewise for n-fold integrals.

Example 2.2

Compute
$$\iiint_{[0,2]\times[1,3]\times[-1,2]} (x^2yz)\ \mathrm{d}V.$$

Solution

$$
\begin{aligned}
\iiint_{[0,2]\times[1,3]\times[-1,2]} (x^2yz)\,\mathrm{d}V &= \int_{-1}^{2}\int_{1}^{3}\int_{0}^{2} (x^2yz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{-1}^{2}\int_{1}^{3}\left[\frac{1}{3}x^3yz\right]_{x=0}^{x=2}\mathrm{d}y\,\mathrm{d}z \\
&= \int_{-1}^{2}\int_{1}^{3}\frac{8}{3}yz\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{-1}^{2}\left[\frac{4}{3}y^2z\right]_{y=1}^{y=3}\mathrm{d}z \\
&= \int_{-1}^{2}(\frac{32}{2})\,\mathrm{d}z \\
&= \left[\frac{16}{3}z^2\right]_{z=-1}^{z=2} = 16.
\end{aligned}
$$

Remark

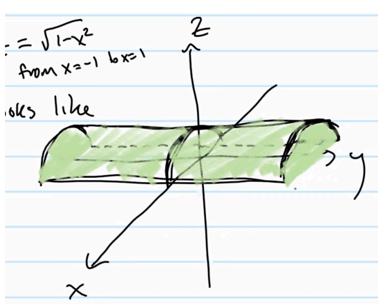
Note that $f(x,y,z) = x^2yz$ is odd with respect to y or z (for all x,z or for all x,y). If y or z is bounded by $[-x,x]$ then the integral would evaluate to 0.

Example 2.3

Some integrals will be easier to evaluate using geometric interpretation. Consider

$$\iint_{[-1,1]\times[2,-2]} \sqrt{1-x^2}\,\mathrm{d}A$$

whose graph looks like $z = \sqrt{1-x^2}$. Then the graph is a half cylinder going along the y-direction.

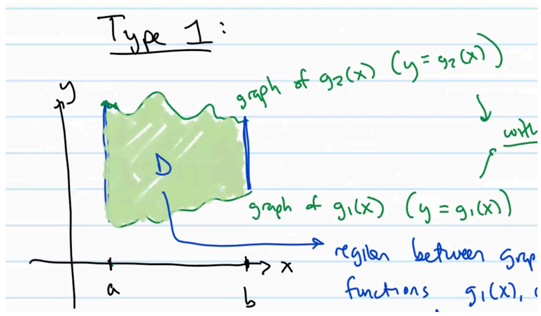


Then the volumn is nothing but $4\cdot\pi/2 = 2\pi$.

3 Fri 10/16 Double Integral over General Shapes

Two kinds of regions:

(1) "Type 1 regions" are defined as the regions between graphs of continuous functions $g_1(x)$ and $g_2(x)$ on the interval $[a, b]$ where $g_1(x) \leqslant g_2(x)$ for $x \in [a, b]$.

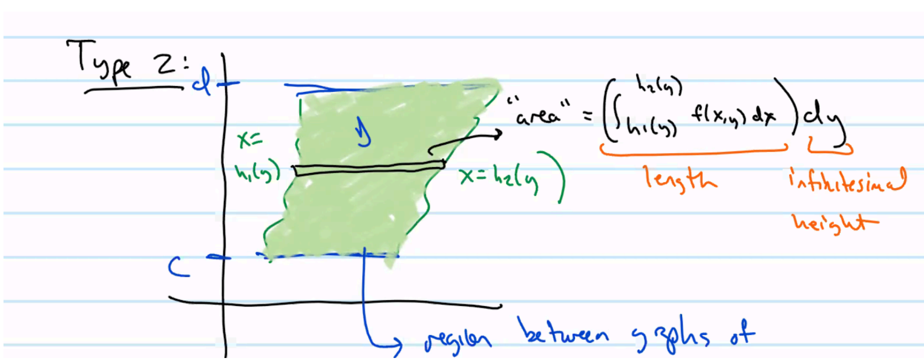


The area can be computed by

$$\iint_D f(x, y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

(2) "Type 2 regions" are just the horizontal version of type 1 regions; for integral, reverse the integral orders.

Some regions can be both type 1 and type 2, for example a box; some can be neither, for example an annuls. Yet the neither case can be expressed as a union of several type 1/2 regions.



3D analogue: a solid between graphs $z = u_1(x, y)$ and $z = u_2(x, y)$ over some domain D in the xy-plane that we can double-integrate over is called type 1. In this case

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}A.$$

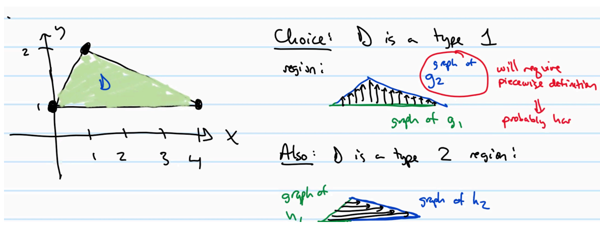
Type 2 solids single out x and type 3 single out y.

Example 3.1

Compute $\iint_D y^2 \, \mathrm{d}A$ where D is the triangular region with vertices $(0,1), (1,2)$, and $(4,1)$.

Solution

Start by drawing D.



Setting D as a type 1 region, requires a piecewise function. Then $g_1(x) = 1$ and $g_2(x)$ is defined as

$$g_2(x) := \begin{cases} x + 1 & \text{for } x \in [0,1] \\ \frac{7}{3} - \frac{x}{3} & \text{for } x \in [1,4] \end{cases}$$

then

$$V = \int_0^1 \int_1^{x+1} y^2 \, \mathrm{d}y \, \mathrm{d}x + \int_1^4 \int_1^{(7-x)/3} y^2 \, \mathrm{d}y \, \mathrm{d}x.$$

Alternatively, we can view D as a type 2 region bounded by $y = 1$ and $y = 2$ in which case we no longer need a piecewise function:

$$V = \int_1^2 \int_{y-1}^{7-3y} y^2 \, \mathrm{d}x \, \mathrm{d}y$$

4 Mon 10/19

Missed first half of class: can think of area of D as $\iint_D 1 \, \mathrm{d}A$ and column of E as $\iiint_E 1 \, \mathrm{d}V$.

★ Don't get confused: $\iiint_E f(x,y,z)\,dV$: a 4-volume.

— only $\boxed{\iiint_E 1\,dV}$ is a (3-) volume.

• e.g. double integral of $f(x,y)$ over a region D doesn't automatically compute the area of D --- only if $f(x,y)$ = const. fn. 1. ($f(x,y)=1$ for all x,y)

Don't forget f!

Example 4.1

Compute the volume of the tetrahedron enclosed by the coordinate planes in $\mathbb{R}^3$ and the plane $2x+y+z=4$.

Solution

We could view it as volume under the graph $z=f(x,y)=4-2x-y$ over the triangle in the xy-plane. This becomes a double integral.

Alternately, we can also think of this as a triple integral over the volume bounded by $u_1(x,y)=0$ and $u_2(x,y)=4-2x-y$ and over the triangle bounded by x-axis, y-axis, and $2x+y=4$. The volume is

$$\begin{aligned}\iint_D \int_0^{4-2x-y} 1 \,\mathrm{d}z\,\mathrm{d}A &= \iint_D 4-2x-y \,\mathrm{d}A \\ &= \iint_D 4-2x-y \,\mathrm{d}x\mathrm{d}y \\ &= \frac{16}{3}\end{aligned}$$

Change of variables

Suppose now we have a new coordinate system (u,v) as opposed to (x,y) where $x=x(u,v)$ and $y=y(u,v)$. Further assume that these functions are invertible. Then D' = image of D under the map sending $(x,y)\mapsto(u(x,y),v(x,y))$. In 1-D case, we have

$$\int_{[a,b]} f(x)\,\mathrm{d}x = \int_{[c,d]} f(x(u))x'(u)\,\mathrm{d}u = \int_{[c,d]} f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u.$$

where $[u(c),u(d)]=[a,b]$. In this case $\frac{\mathrm{d}x}{\mathrm{d}u}$ is the "length-stretching factor". In 2-D and so on, this becomes "local area-stretching factor" and so on.

If we let the transformation be $\phi(u,v) = (x(u,v), y(u,v))$ then the first-order approximation is $J_\phi(u,v)$, which is the image of the unit square under transformation associated with ϕ. This is

$$J_\phi(u,v) = |\det J_\phi(u,v)| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and so

$$\iint_D f(x,y)\, \mathrm{d}x\mathrm{d}y = \iint_{D'} f(u,v) |\det J_\phi|\, \mathrm{d}u\mathrm{d}v.$$

From later notes:

Based on our lecture, if D' is the image of D under the map f defined by $(x,y) \mapsto (u(x,y), v(x,y))$, and $\varphi(u,v) = (x(u,v), y(u,v))$, then $\varphi^{-1} = f$. Therefore if $D' = f(D)$ we have $\varphi(D') = D$. Then

$$\begin{aligned}\iint_D f(x,y)\, \mathrm{d}x\mathrm{d}y &= \iint_{\varphi(D')} f(x,y)\, \mathrm{d}x\mathrm{d}y \\ &= \iint_{\varphi(D')} f(x(u,v), y(u,v)) \underbrace{|J_\varphi(u,v)|}_{\text{stretching factor}} \mathrm{d}x\mathrm{d}y \\ &= \iint_{D'} f(x(u,v), y(u,v)) \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial u/\partial u & \partial y/\partial v \end{vmatrix} \mathrm{d}u\mathrm{d}v.\end{aligned}$$

Notice that this is a generalization of

$$\int_{[a,b]} f(x)\, \mathrm{d}x = \int_{x^{-1}([a,b])} f(x(u)) \frac{\mathrm{d}x}{\mathrm{d}u}\, \mathrm{d}u = \int_{[c,d]} f(x(u)) \frac{\mathrm{d}x}{\mathrm{d}u}\, \mathrm{d}u,$$

where the function $x : [c,d] \to [a,b]$ and $u = x^{-1} : [a,b] \to [c,d]$.

5 Wed 10/21

Example 5.1

Consider the annulus described by $\{(x,y) \mid 1 \leqslant x^2 + y^2 \leqslant 4\}$. Then

$$\varphi(r,\theta) : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

is the mapping from (r,θ) coordinate to (x,y). We have

$$J_\varphi(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

This is where we get $\mathrm{d}x\mathrm{d}y \sim r\mathrm{d}r\mathrm{d}\theta$.

Example 5.2

Let D be the top half of the disk with radius 5 centered at origin. Compute

$$\iint_D x^2 y \, \mathrm{d}x\mathrm{d}y$$

Solution

The area corresponds to $r \in [0,5], \theta \in [0,\pi]$. The integral becomes

$$\iint_D x^2 y \, \mathrm{d}A = \iint_{[0,\pi]\times[0,5]} r \, \mathrm{d}(r^2 \cos^2\theta)(r \sin\theta) r \mathrm{d}\theta = \text{omitted}.$$

Cylindrical Coordinates

Similar to polar coordinates in 2D but with an extra z coordinate.

$$\varphi(r,\theta,z) \mapsto (r\cos\theta, r\sin\theta, z).$$

Here the Jacobian is

$$J_\varphi = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left| \text{top } 2\times 2 \right| = r.$$

Hence $\mathrm{d}x\mathrm{d}y\mathrm{d}z \sim r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$.

Example 5.3

Let E be the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$, Compute $\iiint_E (x+y+z) \, \mathrm{d}V$ in cylindrical coordinates.

Solution

In the first octant, $r \in [0,2], \theta \in [0,\pi/2], z = 4 - x^2 - y^2 = 4 - r^2$. Then

$$\begin{aligned}\iiint_E (x+y+z)\,\mathrm{d}V &= \iint_{E'} (r\cos\theta + r\sin\theta + z)\; r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\ &= \iint_{D'} \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\,\mathrm{d}z\; r\;\mathrm{d}r\mathrm{d}\theta \\ &= \text{omitted.}\end{aligned}$$

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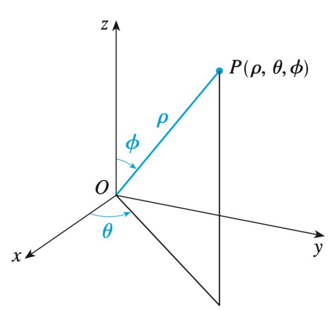
Spherical Coordinates

Another system of coordinates: $(x,y,z) \mapsto (\rho,\theta,\phi)$ with $\rho \in [0,\infty)$, $\theta \in [0,2\pi)$, and $\phi \in [0,\pi]$. Special cases, by convention:

(1) If $\rho = 0$, we define $\theta = \rho = 0$ since this is just the origin.

(2) If $\rho > 0$ and $\rho = 0$ or π, then $\theta = 0$ by convention since this is a point on the z-axis.



In this system, we define

$$\varphi : \begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\cos\phi)\sin\theta \\ z = \rho\cos\phi \end{cases} \quad \text{and } \varphi^{-1} : \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(y/x) \\ \phi = \tan^{-1}(\sqrt{x^2+y^2}/z) \end{cases}$$

The Jacobian determinant of φ is $\rho^2\sin\phi$ [omitted, but similar to r for polar coordinates].

Example 6.1

Compute the volume of E, a ball of radius R in $\mathbb{R}^3$.

Solution

$$\begin{aligned}\iiint_E 1 \text{ d}V &= \iiint_E 1 \cdot \rho^2 \sin\phi \text{ d}\rho\text{d}\theta\text{d}\phi \\ &= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \rho^2 \sin\phi \text{ d}\rho \text{ d}\theta \text{ d}\phi \\ &= \int_0^{R} \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi \text{ d}\phi \text{ d}\theta \text{ d}\rho \\ &= \text{blah blah blah} = \frac{4}{3}\pi R^3.\end{aligned}$$