1 Wed 11/4 Green's, Curl/Divg, Para Surface

Last time: Green's theorem

$$\int_{\partial M} \alpha = \int_M \mathrm{d}\alpha \implies \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C P\, \mathrm{d}x + Q\, \mathrm{d}y = \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, \mathrm{d}A.$$

Example 1.1

Compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} = \langle y - \cos y, x \sin y \rangle$ and $C := \{(x,y) \mid (x-3)^2 + (y+4)^2 = 4\}$, oriented *clockwise*.

Solution

With negative orientation:

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_D 1 + \sin y - \sin y\, \mathrm{d}x \\ &= \iint_D 1\, \mathrm{d}A = 4\pi.\end{aligned}$$

13.5 Curl & Divergence

On $\mathbb{R}^3$: 0-forms are scalar-valued functions; 1-forms are vector fields, and so far 2-forms; 3-forms are scalar-valued functions again. There's a $n - (k - n)$ symmetry. The mappings between forms (n to $n+1$) are called *exterior derivatives*.

Two cases of generalized Stokes':

(1) 1-form to 2-form: α be the 1-form $P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$ where

$$\begin{aligned}\mathrm{d}\alpha &= \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x \\ &\quad \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \\ &\quad \left(\frac{\partial R}{\partial x}\mathrm{d}x + \frac{\partial R}{\partial y}\mathrm{d}y + \frac{\partial R}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x \\ &= \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.\end{aligned}$$

where wedge products like $\mathrm{d}x \wedge \mathrm{d}y$ are called the three *basic 2-forms on* $\mathbb{R}^3$. General 2-form on $\mathbb{R}^3$ can be written as

$$A(x,y,z)\mathrm{d}y \wedge \mathrm{d}z + B(x,y,z)\mathrm{d}z \wedge \mathrm{d}x + C(x,y,z)\mathrm{d}x \wedge \mathrm{d}y$$

which we can view as the vector field

$$\langle A(x,y,z), B(x,y,z), C(x,y,z)\rangle .$$

Therefore (for $\mathbb{R}^3$ only), the exterior derivative serves as an operator that transforms vector fields into vector fields:

Definition 1

The **curl** operator sends $\mathbf{F}$ on $\mathbb{R}^3$ to another vector field:

$$\langle P,Q,R\rangle \mapsto \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle .$$

We denote curl of $\mathbf{F}$ as curl $\mathbf{F}$ or $\nabla \times \mathbf{F}$ thanks to the "fake determinant":

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \operatorname{curl} \mathbf{F}.$$

Example 1.2

Let $\mathbf{F} = \langle e^x \sin y, e^y \sin z, e^z \sin x\rangle$. Find its curl.

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^y \sin z & e^z \sin x \end{vmatrix} = -\langle e^y \cos z, e^z \cos x, e^x \cos y\rangle .$$

Remark

Geometrically, curl measures the direction *and* magnitude of "curliness" of vector field near each point. "Curl the fingers" of right hand, and the direction where the thumb points toward is the direction of curl. Curl appears nonzero where there are little whirlpools. On the other hand, a vector field is called **irrotational** if its curl vanishes everywhere.

(2) 2-form to 3-form: if we view $\mathbf{F} = \langle P,Q,R\rangle$ on $\mathbb{R}^3$ as

$$P\mathrm{d}y \wedge \mathrm{d}z + Q\mathrm{d}z \wedge \mathrm{d}x + R\mathrm{d}x \wedge \mathrm{d}y$$

then its external derivative (something like $\left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z + \dots$) would be

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

and if we translate this back to the function, we have

Definition 2

The **divergence** of a vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

which we also denote as $\nabla \cdot \mathbf{F}$ for obvious reasons. :)