1 Fri 10/23

13.1 Vector Fields

Definition 1

A **vector field** is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ that assigns each point to a vector in $\mathbb{R}^n$.

Remark

In this chapter we will refer to $\mathbf{F}$ itself as a vector field on $\mathbb{R}^n$.

Example 1.1

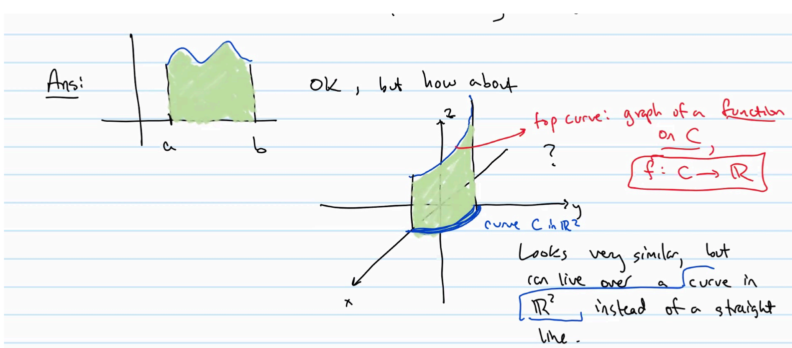
If $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar valued function, then ∇f is a vector field on $\mathbb{R}^n$. (Each point in $\mathbb{R}^n$ is assigned ∇ of that point, another vector in $\mathbb{R}^n$.)

Definition 2

A vector field $\mathbf{F}$ on $\mathbb{R}^n$ is called a **conservative vector field** if it is of the form ∇f for some scalar-valued function f. If this is the case, we call f a **potential function** for $\mathbf{F}$.

13.2 Line Integrals

Previously single-variable calculus defines how to compute the area under a curve on some interval $[a, b]$ which can visualized as a straight line. Now think of a curve on xy-plane and a function defined above that curve.



This will be

$$\int_C f \, \mathrm{d}s,$$

the line integral of f on C with respect to arc-length. This will be the first type of line integral for us.

A preview: think of a particle moving in $\mathbb{R}^3$ such that it's subject to force $\mathbf{F}(x, y, z)$ at (x, y, z). If we move the particle along the path given by $\mathbf{r}(t), a \leqslant t \leqslant b$, then the work done by the field will be

$$\int_C (\mathbf{F} \cdot \mathbf{T}) \, \mathrm{d}s = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P(x,y,z) \, \mathrm{d}x + Q(x,y,z)\mathrm{d}y + R(x,y,z)\mathrm{d}z \text{ where } \mathbf{F} = \langle P, Q, R \rangle .$$

Let C be a curve in $\mathbb{R}^2$, parametrized by $\mathbf{r} : [a, b] \to \mathbb{R}^2$. Let $f : C \to \mathbb{R}$ be a scalar valued function on C [can think of this as $f : \mathbb{R} \to \mathbb{R}$ restricted on C]. Then the line integral is the sum of "infinitesimal area" of base times height: $(\|\mathbf{r}'(t)\|\mathrm{d}t)(f(\mathbf{r}(t))$. [The base is expressed in terms of t so the "velocity factor" $\|\mathbf{r}'(t)\|$ matters.] Then the "area under the graph" is

$$\int_C f \, \mathrm{d}s = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\| \, \mathrm{d}t = \int_a^b f(x(t), y(t))\sqrt{(\mathrm{d}x/\mathrm{d}t)^2 + (\mathrm{d}x/\mathrm{d}y)}^2 \, \mathrm{d}t.$$

2 10/28 Line Integrals, FT of Line integrals

Last time we covered two types of integrals:

$$\int_C f \, \mathrm{d}s \text{ and } \int_C (\mathbf{F} \cdot \mathbf{T}) \, \mathrm{d}s = \int_C \mathbf{F} \, \mathrm{d}\mathbf{r} \text{ [if parametrized by } \mathbf{r}],$$

the first called the line integral of f on C with respect to arclength and the second the line integral of a vector field $\mathbf{F}$ on C. The second one depends on curve but *not* on $\mathbf{r}$, i.e., independent of parametrization.

Proposition 3

A parametrization implies an orientation. If we are given $\mathbf{r}$, we don't need to think hard about the parametrization, but if we are not given $\mathbf{r}$, careful. This will be similar for surface integrals.

Now we get to a third notation for line integrals of vector fields (focusing on $\mathbb{R}^2$ for now):

Proposition 4

If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$, a vector field on $\mathbb{R}$, then the line integral of this vector field on C is given by

$$\int_C (\mathbf{F} \cdot \mathbf{T}) \, \mathrm{d}s = \int_C \mathbf{F} \, \mathrm{d}\mathbf{r} = \int_C P(x,y) \, \mathrm{d}x + \int_C Q(x,y) \, \mathrm{d}y.$$

Remark

Recall that if C is parametrized by $\mathbf{r}(t) = (x(t), y(t))$ for $t \in [a,b]$, then

$$\begin{aligned}\int_C \mathbf{F}\ \mathrm{d}\mathbf{r} &= \int_a^b [P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)]\ \mathrm{d}t \\ &= \int_a^b P(x,y)\ \mathrm{d}x + \int_a^b Q(x,y)\ \mathrm{d}y.\end{aligned}$$

Behind the scenes: $P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$ is the notation for a *differential 1-form* on $\mathbb{R}^2$ that corresponds to a vector field on $\mathbb{R}^2$, namely $\mathbf{F} = \langle P(x,y), Q(x,y)\rangle$. This allows you to integrate a "differential 1-form on C". Integrals defined using "pullback" of differential form by a parametrization $\mathbf{r}$ of C is precisely what we get:

$$\mathbf{r}^*(P\mathrm{d}x + Q\mathrm{d}y) = [P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)]\mathrm{d}t.$$

$\mathrm{d}s$ is also a differential 1-form on C, and its pullback in a parametrization $\mathbf{r}$ is $\mathbf{r}^*(\mathrm{d}s) = \|\mathbf{r}'(t)\|\mathrm{d}t$. Then we get the familiar

$$\int_C f\ \mathrm{d}s = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ \mathrm{d}t.$$

If $P\mathrm{d}x + Q\mathrm{d}y$ is a differential 1-form on $\mathbb{R}^2$, it gives a differential 1-form on C defined by $(\mathbf{F}\cdot\mathbf{T})\mathrm{d}s$ where $\mathbf{F} = \langle P, Q\rangle$. This is why $\int_C (\mathbf{F}\cdot\mathbf{T})\ \mathrm{d}s$ is so natural.

Fundamental Theorem of Calculus for Line Integrals

This is a special case of the "generalized Stokes' theorem" for differential forms (so are Green's, classical Stokes', Divergence).

For calc III we'd be integrating "d-manifolds in $\mathbb{R}^n$": $d = 1$ represents curves, $d = 2$ represents regions, $d = 3$ represents surfaces, and so on.

Generalized Stokes' Theorem

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha.$$

Here M is a d-manifold in $\mathbb{R}^n$, ∂M the boundary of M, a (d-1)-manifold in $\mathbb{R}^n$, α some differential (d-1)-form on M, and $\mathrm{d}\alpha$ the "exterior derivative" of α.

Example 2.1

Let M be a curve in $\mathbb{R}^n$, a case with $d = 1$, and α a scalar-valued function f on C (a differential 0-form on

C). Then $\mathrm{d}\alpha$ is the 1-form on $\mathbb{R}^2$ written as

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

which corresponds to the gradient vector field ∇f on $\mathbb{R}^2$. Let $M = C$ a curve and so ∂M are two "signed" points: a positive ending point of the curve and a negative starting point of curve. Then

$$\int_M \mathrm{d}\alpha = \int_C \left(\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y\right) = \int_C (\nabla f)\cdot \mathrm{d}\mathbf{r}$$

and

$$\int_{\partial M} \alpha = \int_{\partial C} f = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

The two sides give

Fundamental Theorem for Line Integrals

Let C be a smooth curve parametrized by $\mathbf{r}(t), t \in [a,b]$ and f be a differentiable function of two variables. Then

$$\int_C (\nabla f)\cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)),$$

Remark

This gives rise to the fact that $\int \mathbf{F}\cdot\ \mathrm{d}\mathbf{r}$ is independent of choice of path as long as the curves have the same starting point and the same ending point.

3 Mon 11/2

Recall from last time that

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha.$$

and the special case FT of LI:

$$\int_C (\nabla \mathbf{F})\cdot\ \mathrm{d}\mathbf{r} = \mathbf{F}(t_2) - \mathbf{F}(t_1).$$

Also recall that integrals like $\int_C (\nabla \mathbf{F})\cdot \mathrm{d}\mathbf{r}$ are independent of not only the parametrization but also the choice of path.

Theorem 7

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\mathbf{F}$ is conservative in D. In other words, the line integral of every closed path evaluates to 0.

Remark

To show $\mathbf{F}$ is not conservative, it suffices to find the existence of one closed curve whose line integral evaluates to a nonzero value. To show $\mathbf{F}$ is conservative, we need a better approach (clearly we can't enumerate *all* closed curves).

Definition 8

Suppose a vector field $\mathbf{F} \in \mathbb{R}^2$ is given by $\mathbf{F} = \langle P(x,y), Q(x,y) \rangle$. If $\mathbf{F}$ is the gradient of some scalar-valued function, then

$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y).$$

Problem 1

If the equation above holds for all (x,y) in the domain of $\mathbf{F}$, can we conclude that $\mathbf{F}$ is conservative?

Solution

Depends on the topology of the domain. It's always true of the domain is simply connected. For multiply connected regions this may fail.

Example 3.1

Let $\mathbf{F}(x,y) = \langle e^x \sin y, e^x \cos y \rangle$ a vector field on all of $\mathbb{R}^2$. Determine if $\mathbf{F}$ is conservative. If so, find the potential function to which $\mathbf{F}$ is the gradient.

Solution

A quick check:

$$\frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y; \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y.$$

Hence $\mathbf{F}$ is conservative (since the domain is simply connected). Now for the potential function:

$$f(x,y) = \int e^x \sin y \, \mathrm{d}x = e^x \sin y + C(y).$$

Differentiating with respect to y gives

$$e^x \cos y + C'(y) = e^x \cos y \implies C'(y) = 0$$

Therefore $C(y) = c$ and the potential function is given by

$$f(x,y) = e^x \sin y + c.$$

Behind the scenes: if $\alpha = P\mathrm{d}x + Q\mathrm{d}y$ then

$$\begin{aligned} \mathrm{d}\alpha := & \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y\right) \wedge \mathrm{d}y \\ &= \frac{\partial P}{\partial y}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial Q}{\partial x}\mathrm{d}x \wedge \mathrm{d}y \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

since $\mathrm{d}x \wedge \mathrm{d}x = 0$ and $-\mathrm{d}y \wedge \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y$.

Remark

On a simply connected domain, the differential form is exact if and only if it's closed.

Now recall Stokes' again:

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha.$$

If we define M as a closed region whose boundary is C (with positive orientation) and α a differential form $P\mathrm{d}x+Q\mathrm{d}y$ whose exterior derivative is

$$\mathrm{d}\alpha = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A$$

as mentioned above, we get Green's theorem:

Green's Theorem

With the assumptions above,

$$\oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A.$$

4 2020/11/4 Green's, Curl/Divg, & Param Surface

Last time: Green's theorem

$$\int_{\partial M} \alpha = \int_M \mathrm{d}\alpha \implies \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A.$$

Example 4.1

Compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} = \langle y - \cos y, x \sin y\rangle$ and $C := \{(x, y) \mid (x-3)^2 + (y+4)^2 = 4\}$, oriented *clockwise*.

Solution

With negative orientation:

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_D 1 + \sin y - \sin y\,\mathrm{d}x \\ &= \iint_D 1\,\mathrm{d}A = 4\pi.\end{aligned}$$

13.5 Curl & Divergence

On $\mathbb{R}^3$: 0-forms are scalar-valued functions; 1-forms are vector fields, and so far 2-forms; 3-forms are scalar-valued functions again. There's a $n - (k - n)$ symmetry. The mappings between forms (n to $n + 1$) are called *exterior derivatives*.

Two cases of generalized Stokes':

(1) 1-form to 2-form: let α be the 1-form $P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$. Then d acts on α by

$$\begin{aligned}
\mathrm{d}\alpha = &\left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\mathrm{d}z\right)\wedge \mathrm{d}x \\
&+\left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z\right)\wedge \mathrm{d}y \\
&+\left(\frac{\partial R}{\partial x}\mathrm{d}x + \frac{\partial R}{\partial y}\mathrm{d}y + \frac{\partial R}{\partial z}\mathrm{d}z\right)\wedge \mathrm{d}z \\
=& \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\wedge\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\wedge\mathrm{d}x \\
=& \operatorname{curl}\mathbf{F}\cdot\mathrm{d}\mathbf{S}.
\end{aligned}$$

The wedge products like $\mathrm{d}x \wedge \mathrm{d}y$ are called the three *basic 2-forms on* $\mathbb{R}^3$. General 2-forms on $\mathbb{R}^3$ can be written as

$$A(x,y,z)\mathrm{d}y\wedge\mathrm{d}z + B(x,y,z)\mathrm{d}z\wedge\mathrm{d}x + C(x,y,z)\mathrm{d}x\wedge\mathrm{d}y$$

which we can view as the vector field

$$\langle A(x,y,z), B(x,y,z), C(x,y,z)\rangle.$$

Therefore (for $\mathbb{R}^3$ only), the exterior derivative serves as an operator that transforms vector fields into vector fields:

Definition 10

The **curl** operator sends $\mathbf{F}$ on $\mathbb{R}^3$ to another vector field:

$$\langle P, Q, R\rangle \mapsto \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle.$$

We denote curl of $\mathbf{F}$ as curl $\mathbf{F}$ or $\nabla\times\mathbf{F}$ thanks to the "fake determinant":

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \operatorname{curl}\mathbf{F}.$$

Example 4.2

Let $\mathbf{F} = \langle e^x\sin y, e^y \sin z, e^z\sin x\rangle$. Find its curl.

Solution

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^y \sin z & e^z \sin x \end{vmatrix} = -\langle e^y \cos z, e^z \cos x, e^x \cos y \rangle.$$

Remark

Geometrically, curl measures the direction *and* magnitude of "curliness" of vector field near each point. "Curl the fingers" of right hand, and the direction where the thumb points toward is the direction of curl. Curl appears nonzero where there are little whirlpools. On the other hand, a vector field is called **irrotational** if its curl vanishes everywhere.

(2) 2-form to 3-form: if we view $\mathbf{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ as

$$P\mathrm{d}y \wedge \mathrm{d}z + Q\mathrm{d}z \wedge \mathrm{d}x + R\mathrm{d}x \wedge \mathrm{d}y$$

then its external derivative (something like $\left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z + \dots$) would be

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

and if we translate this back to the function, we have

Definition 11

The **divergence** of a vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

which we also denote as $\nabla \cdot \mathbf{F}$ for obvious reasons. :)

5 Fri 11/6

Quick summary of different operators:

(1) Gradient: $f(x, y, z) \mapsto \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$, scalar-valued function to vector field.

(2) Curl ($\mathbb{R}^3$ only): $\mathbf{F} := \langle P, Q, R \rangle \mapsto \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$, vector field to vector field.

(3) Divergence: $\mathbf{F} := \langle P, Q, R\rangle \mapsto \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$, vector field to scalar.

(4) Laplacian: $f(x, y, z) \mapsto \nabla^2\mathbf{F} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$, scalar-valued function to scalar.

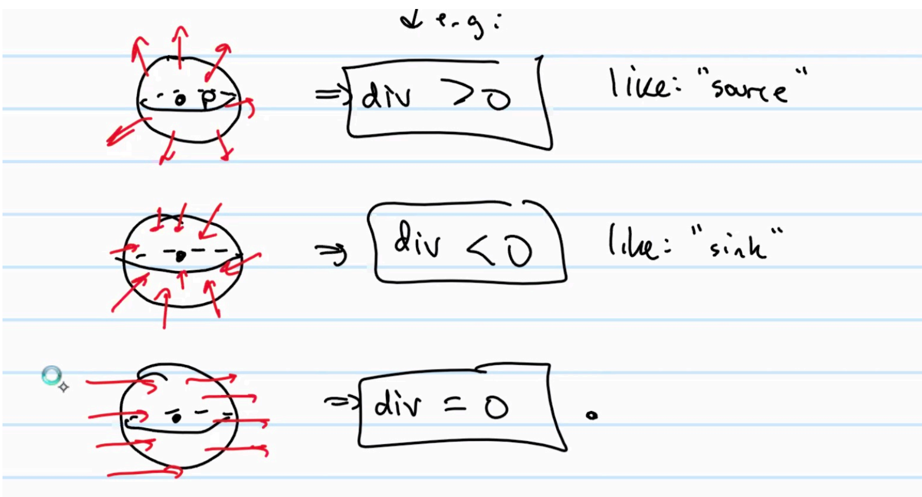
Example 5.1

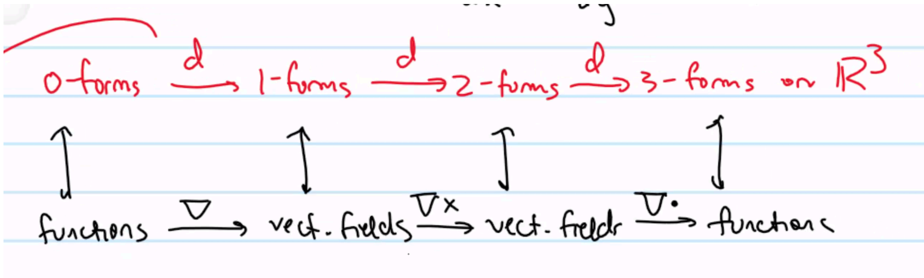
Compute the divergence of $\mathbf{F} := \langle e^x \sin y, e^y \sin z, e^z \sin x\rangle$.

Solution

$$\text{div } \mathbf{F} = e^x \sin y + e^y \sin z + e^z \sin x.$$

Geometric & physical interpretation of divergence: it measures the "source/sink" nature of the vector field at a point. Source if divergence > 0; sink if divergence < 0. Like how we call $\mathbf{F}$ irrotational if curl $\mathbf{F} = 0$ wherewhere, we cay $\mathbf{F}$ **incompressible** if div $\mathbf{F} = 0$ everywhere.





Very interesting "fact": $d^2 = d \circ d = 0$: if we do any two the following in a row we get 0:

$$\text{function} \xrightarrow{\text{gradient}} \text{vector field} \xrightarrow{\text{curl}} \text{vector field} \xrightarrow{\text{divergence}} \text{function}$$

Theorem 12

Let $\mathbf{F} := \langle P, Q, R \rangle$, a vector field on $\mathbb{R}^3$. If $\mathbf{F}$ is conservative then curl $\mathbf{F} = 0$.

The converse depends on the topology of the domain. De Rham's theorem, for regions E in $\mathbb{R}^3$ that are simply connected ("don't link any circle"), then the converse is true. If this is the case, curl $\mathbf{F} = 0$ if and only if $\mathbf{F}$ is conservative.

Remark

Is $\mathbf{F}$ the curl of any vector field? If so, the divergence of $\mathbf{F}$ must be 0.

Converse: if the domain "doesn't trap a fly", this becomes a iff.

Definition 13

But what if we skip the curl and take the gradient of divergence? Then we get the **Laplacian** of a function $f(x, y, z)$, denoted as Δf.

$$\Delta f = \nabla \cdot \nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

One final remark about $\nabla, \nabla\times$, and $\nabla\cdot$: our formulas are tied to Cartesian coordinates. Suppose now we are in polar coordinates, then things get weird.

$$\nabla f(r, \theta) = \left\langle \frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta} \right\rangle.$$

In $\mathbb{R}^3$ cylindrical and spherical, these become

$$\nabla f(r, \theta, z) = \left\langle \frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right\rangle \text{ and } \nabla f(\rho, \theta, \phi) = \left\langle \frac{\partial f}{\partial \rho}, \frac{1}{\rho \sin\phi}\frac{\partial f}{\partial \theta}, \frac{1}{\rho}\frac{\partial f}{\partial \phi} \right\rangle.$$

This provides an advantage of $\mathrm{d}f$ over ∇f since it's a lot simpler to write down.

6 Mon 11/9 Param. Surfaces & Surface Integrals

Similar to line integrals, surface integrals $\iint_S \dots dudv$ are computed by parametrizing a surface S in 2d "uv" surface where S is the *image* of $\mathbf{r}(u,v)$ on domain D.

Example 6.1

(1) A ball in $\mathbb{R}^3$ with radius 2 centered at $(1,2,3)$: with spherical coordinates we have

$$\mathbf{r}(\phi,\theta) := \langle 1,2,3\rangle + 2\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle .$$

(2) Plane in $\mathbb{R}^3$ containing (x_0,y_0,z_0) and the vectors $\mathbf{a},\mathbf{b}$:

$$\mathbf{r}(u,v) = \langle x_0,y_0,z_0\rangle + u\mathbf{a} + v\mathbf{b},\ (u,v)\in\mathbb{R}^2.$$

(3) *** Surface S in $\mathbb{R}^3$ which is the graph of a function (WLOG) $z = f(x,y)$:

$$\mathbf{r}(u,v) := \langle u, v, f(u,v)\rangle \text{ for } (u,v)\in D.$$

Likewise if x or y becomes the dependent variable.

Definition 14

The unit normal vector to a surface S by a parametrization $\mathbf{r}(u,v)$ is

$$\mathbf{n}(\mathbf{r}(u,v)) = \frac{\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}}{\left\|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right\|}.$$

Compare this with $\mathbf{T}(\mathbf{r}(t)) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$. But why? The partial derivatives at a point (u,v) are the velocity vectors of curves in the surface. Taking the cross product gives a vector that's normal to the tangent plane at (u,v). Normalizing it gives the **unit normal vector**. By convention we choose this over the negative one as the unit normal vector.

Example 6.2: 13.7.23 (Stewart)

Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ above the square $[0,1]\times[0,1]$ with upward orientation (i.e., unit normal vectors pointing upwards). This surface can be parametrized by

$$\mathbf{r}(u,v) := \left\langle u, v, 4-u^2-v^2\right\rangle \text{ for } (u,v)\in[0,1]\times[0,1].$$

Then a normal vector at (u, v) is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2u \\ 1 & 0 & -2v \end{vmatrix} = \langle 2u, 2v, 1 \rangle \implies \mathbf{n}(\mathbf{r}(u,v)) = \frac{1}{\sqrt{4u^2 + 4v^2 + 1}} \langle 2u, 2v, 1 \rangle .$$

(From the normal vector we already see that this parametrization gives unit normal vectors pointing upwards.)

Remark

If we end up getting the opposite orientation, simply swap u and v and we'll get the desired orientation. (Keep in mind that when we swap u and v, the domain gets "flipped" as well.)

Example 6.3: 13.6.31 (Stewart)

Let S be parametrized by $\mathbf{r}(u,v) = \langle u \cos v, u \sin v, v \rangle$. Find equation to the tangent plane to S at point $p = \mathbf{r}(1, \pi/3) = (1/2, \sqrt{3}/2, \pi/3)$.

Solution

We know p is on the plane. Now we need to get a normal vector (can be normalized but not necessary):

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \sin v & 1 \end{vmatrix} = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 1 \right\rangle .$$

Therefore the plane can be expressed as the equation

$$\frac{\sqrt{3}}{2}(x - 0.5) - \frac{1}{2}(x - \frac{\sqrt{3}}{2}) + (x - \frac{\pi}{3}) = 0.$$

Definition 15

If a surface is parametrized by $\mathbf{r}(u,v)$ for $(u,v) \in D$ then the surface area of S is

$$\iint_D \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \mathrm{d}A.$$

Compare this to the parametrized arc length

$$\int_{t_1}^{t_2} |r'(t)| \, \mathrm{d}t.$$

Example 6.4: 13.6.35

Let S be the part of the plane $x + 2y + 3z = 1$ that lies in the cylinder $x^2 + y^2 = 3$ in $\mathbb{R}^3$.

Solution

S can also be written as the graph of $z(x, y) = (1 - x - 2y)/3$. Then

$$\text{Area of } S = \iint_D \| \langle 1, 0, -1/3 \rangle \times \langle 0, 1, -2/3 \rangle \| \, \mathrm{d}A = \iint_D \sqrt{14}/3 \, \mathrm{d}A = \sqrt{14}\pi.$$

7 Wed 11/11

Recall that, just like the line integral on C is independent of the parametrization $\mathbf{r}(t)$, the surface integral of surface S is independent of $\mathbf{r}(u, v)$.

$$\begin{cases} \int_C f(x,y) \, \mathrm{d}s = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, \mathrm{d}t \\ \int_S f(x,y,z) \, \mathrm{d}a = \iint_D f(\mathbf{r}(u,v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, \mathrm{d}A. \end{cases}$$

Example 7.1: 13.7.7 (Stewart)

Compute $\iint_S y \, \mathrm{d}S$ where S is the "helicoid": image of

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, u \in [0, 1], v \in [0, \pi].$$

Solution

This would be

$$\iint_D u \sin v \| \sim \| \, \mathrm{d}v.$$

Norm of the "stretching factor" is

$$\begin{vmatrix} \mathbf{j} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

Then

$$\iint_D u \sin v \sqrt{1+u^2} \, \mathrm{d}A = \int_0^\pi \int_0^1 u \sin v \sqrt{1+u^2} \, \mathrm{d}u \, \mathrm{d}v = \text{ omitted.}$$

How about integrating vector fields on surfaces, just like integrating vector fields along lines (works done by a vector field)?

Definition 16

If $\mathbf{F}$ is a continuous vector field defined on an *oriented* surface S with a unit normal vector $\mathbf{n}$, then the **flux** or **surface integral of F** is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S.$$

Remark

Think of this is "total amount of flow on a surface".

This simplifies when computing a parametrization: say S is parametrized by $\mathbf{r}(u,v)$ on a domain D. Then

$$\mathbf{n}(\mathbf{r}(u,v)) = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \|}$$

whereas

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \|} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \mathrm{d}x = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \mathrm{d}x$$

which the book calls

$$\iint_S \mathbf{F} \cdot \, \mathrm{d}\mathbf{S}.$$

Relate this with

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t.$$

Example 7.2: 13.7.23 (Stewart

Compute $\iint_S \mathbf{F} \cdot \,\mathrm{d}\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = \langle xy, yz, zx \rangle$$

and S is the part of paraboloid $z = 4 - x^2 - y^2$ lying above the square $[0,1] \times [0,1]$ with upward orientation.

Solution

From last time:

$$\mathbf{r}(u, v) = \left\langle u, v, 4 - u^2 - v^2 \right\rangle, (u, v) \in [0,1] \times [0,1].$$

Then

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1 \rangle.$$

Therefore

$$\iint_S \mathbf{F} \cdot \,\mathrm{d}\mathbf{S} = \iint_{[0,1]\times[0,1]} \left\langle uv, v(4 - u^2 - v^2), u(4 - v^2 - v^2) \right\rangle \,\mathrm{d}u\mathrm{d}v = \text{ omitted.}$$

Remark

If $\mathbf{F}$ is the curl of some $\mathbf{G}$, i.e., $\mathbf{F} = \nabla \times \mathbf{G}$ then $\iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S$ depends only on the boundary curve of S, not the rest. Compare this with the fundamental theorem of line integrals.

Furthermore, with the assumption above, if S is a closed surface (no boundary curve), then

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = 0, \text{ also analogous to a closed curve and its line integral.}$$

Preview: (classical) Stokes' theorem; divergence theorem. Really just two special cases of generalized Stokes'.

8 Fri 11/13 Stokes's & Divergence Thm

Stokes's Theorem (Classical)

Theorem 17

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and S an oriented surface with its induced oriented boundary C (with right-hand rule: thumb pointing toward the diction of normal vectors), then

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S.$$

Example 8.1: 13.8.7 (Stewart)

Use Stokes's theorem to compute $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ where

$$\mathbf{F} = \langle yz, 2xz, e^{xy}\rangle \text{ and } C \text{ is the circle } x^2+y^2=16, z=5, \text{ oriented counterclockwise.}$$

Solution

Let S be the surface $x^2+y^2\leqslant 16, z=5$, oriented up. Then a way to parametrize S can be given by

$$\mathbf{r}(u,v) = (u,v,5) \text{ for } (u,v)\in\{(u,v)\mid u^2+v^2\leqslant 16\}.$$

Then

$$\begin{cases} \frac{\partial \mathbf{r}}{\partial u} = \langle 1,0,0\rangle \\ \frac{\partial \mathbf{r}}{\partial v} = \langle 0,1,0\rangle \end{cases} \implies \left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\| = \|\langle 0,0,1\rangle\| = 1.$$

On the other hand,

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix} = \langle xe^{xy}-2x, y-ye^{xy}, z\rangle$$

and so

$$\begin{aligned} \oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S} \\ &= \iint_D \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\|\langle 0,0,1\rangle\|\,\mathrm{d}S \\ &= \iint_D 5\,\mathrm{d}u\mathrm{d}v = 80\pi. \end{aligned}$$

Conceptually: $\oint_C \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$ for any S in $\mathbb{R}^3$, given $\mathbf{F}$:

(1) Take S, a very small disk, near some point P, with normal vector in some direction $\mathbf{n}$.

(2) Since P is very small, curl $\mathbf{F}$ is almost constant on the disk. Then curl $\mathbf{F} \cdot \mathbf{n}$ is also almost a constant.

(3) Then $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S \sim$ the value of curl $\mathbf{F} \cdot \mathbf{n}$ at P times the area of the small disk S.

(4) On the other hand, by Stokes's this is also $\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

(5) So if $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is positive everywhere on the boundary of a disk, $\mathbf{F}$ does work on the disk and so it "spins", thus providing a positive curl.

Divergence Theorem

Theorem 18

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and E a region in $\mathbb{R}^3$ with boundary S (with orientation given by the outward-pointing normal vector with respect to E), then

$$\oiint_S (\mathbf{F} \cdot \mathbf{n}) \mathrm{d}S = \oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E \text{div } \mathbf{F} \, \mathrm{d}V.$$

Connect this with generalized Stokes': curl $\mathbf{F}$ the 2-form and div $\mathbf{F}$ the 3-form.

Example 8.2

Compute the flux of $\mathbf{F}$ across S where

$$\mathbf{F} = \left\langle x^2 \sin y, x \cos y, -xz \sin y \right\rangle$$

and S given by the equation $x^8 + y^8 + z^8 = 8$.

Solution

The divergence of $\mathbf{F} = 2x \sin y - x \sin y - x \sin y = 0$. Therefore

$$\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E 0 \, \mathrm{d}V = 0.$$

Conceptually:

(1) Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$.

(2) For any tiny ball E near point P, the divergence is almost a constant and so

$$\iiint_E \text{div } \mathbf{F} \, \mathrm{d}V \sim \text{div } \mathbf{F} \text{ at } P \cdot \text{volume of } E.$$

(3) Therefore, by Divgercence theorem,

$$\operatorname{div} \mathbf{F}(P) \sim \frac{\oiint_{\partial E} \mathbf{F} \cdot \mathrm{d}\mathbf{S}}{\text{volume of } E}.$$

(4) If $\mathbf{F} \cdot \mathbf{n} > 0$ everywhere then the divergence here is also positive, a "source". If all negative then a "sink".