MATH 425A HOMEWORK# 1

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Problem 1 (1.1)**.** Prove that for all sets A, B, C the formula

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

is true.

Proof. To prove the LHS (left hand side) = the RHS (right hand side), we need to show both (LHS $\subset$ RHS) and (RHS $\supset$ LHS), and both proofs are analagous to the one given in *example* 0.

To show $\subset$, suppose $x \in A \cup (B \cap C)$. Then it follows that $x \in A$ or $x \in B \cap C$. In other words, x belongs to A or x belongs to both B and C. (Note that the "or" here is the mathematical "or" — both propositions can be simultaneously true.) Now look at x. Like previously said, there are two possibilities: $x \in A$ or $x \in B \cap C$.

(1) If $x \in A$ then it belongs to both $A \cup B$ and $A \cup C$, so it belongs to $(A \cup B) \cap (A \cup C)$.

(2) If $x \in B \cap C$ then it belongs to both B and C and thus both $A \cup B$ and $A \cup C$, i.e., $x \in (A \cup B) \cap (A \cup C)$.

We have just shown that every element in the LHS is also in the RHS, namely LHS $\subset$ RHS.

Now for $\supset$. Suppose $x \in (A \cup B) \cap (A \cup C)$. It follows that $x \in A \cup B$ and $x \in A \cup C$. Again we have two cases, either $x \in A$ or $x \notin A$. The first case immediately leads to $x \in A \cup (B \cap C)$. In the latter case x must belong to both B and C, i.e., $x \in B \cap C$, to meet the requirement. Therefore no matter where x is, we always have $x \in A \cup (B \cap C)$. Thus LHS $\supset$ RHS.

Since LHS $\subset$ RHS and RHS $\supset$ LHS, we conclude that these two sets are indeed equal. ☐

Problem 2 (1.2)**.** If the sets $A, B, C, \ldots$ are subsets of the same set X then the differences $X \setminus A, X \setminus B, X \setminus C, \ldots$ are the **complements** of $A, B, C, \ldots$ in X and are denoted $A^c, B^c, C^c, \ldots$. The symbol A^c is read "A complement".

(1) Prove that $(A^c)^c = A$.

(2) Prove **De Morgan's Law**: $(A \cap B)^c = A^c \cup B^c$ and derive from it the law $(A \cup B)^c = A^c \cap B^c$.

(3) Draw Venn diagrams to illustrate the two laws.

(4) Generalize these laws to more than two sets.

Solution. I hate drawing diagrams in LaTeX and I'm bad at using Inkscape...

(1) $(A^c)^c = \{x \in X \mid x \text{ is not in the set of all elements not included in } A\}$
$= \{x \in X \mid x \text{ is not an element not included in } A\} = \{x \in X \mid x \in A\} = A.$

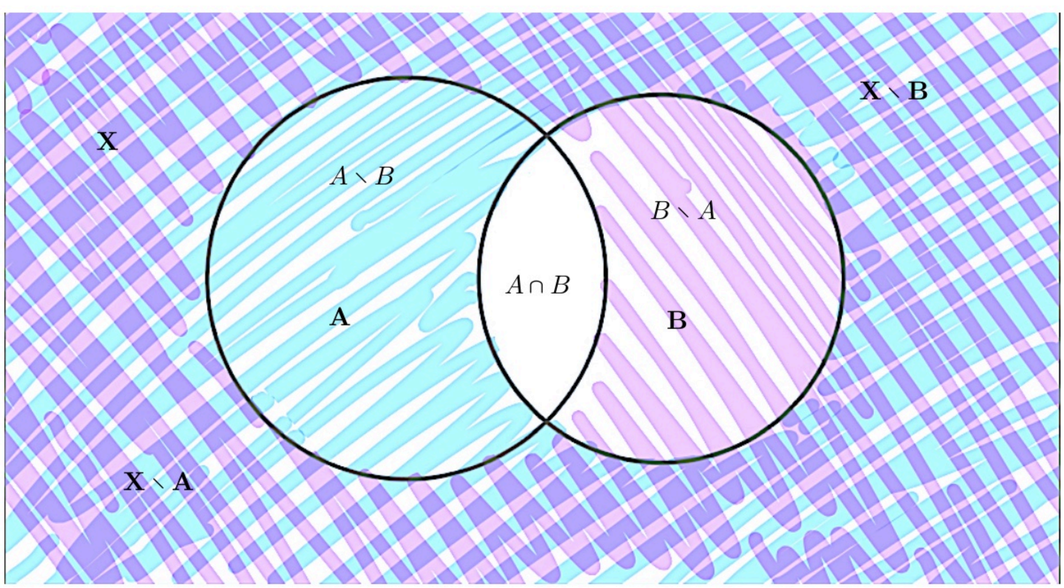
(2) A quick proof: $x \in (A \cap B)^c \iff x \notin A \cap B \iff x \notin A \vee x \notin B \iff x \in A^c \vee x \in B^c \iff x \in A^c \cup B^c$.

A derivation:

$$
\begin{aligned}
& (A^c \cap B^c)^c = (A^c)^c \cup (D^c)^c && \text{(By applying De Morgan's law to } A^c \text{ and } B^c\text{)} \\
\implies & [(A^c \cap B^c)^c]^c = [(A^c)^c \cup (D^c)^c]^c && \text{(Taking the complement of both sides, the equation still holds)} \\
\implies & A^c \cap B^c = (A \cup B)^c && \text{(By the result from (1), complement of complement equals self)}
\end{aligned}
$$

† Jiayue reminded me that the question asks me to derive the second statement directly from the first (De Morgan's law), while I originally proved the second one without referring to De Morgan's law:w. Credits to her.

(3) The entire rectangle is the set X. The circle on the left represents A and the one on the right reppresents B. Both sides of the first equation is represented by all the regions that have been colored, and both sides of the second equation are represented by the region in both purple and cyan. But seriously drawing — especially coloring is 10 times harder than the problem itself...



(4) Generalized De Morgan's law: suppose $A_1, A_2, \ldots, A_n \in X$, then

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c \text{ and } \left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c.$$

This generalized result can be easily proven by weak induction: let $\varphi(k)$ be the statement that De Morgan's law holds when $n = k$. $\varphi(1)$ is trivial. Set the base case to $k = 2$ and we see that $\varphi(2)$ is indeed true as proven in (2). For the inductive step, first assume $\varphi(m)$ is true. Then if we set $n = m$ and let everything on one side of the big equation above be one set and A_{m+1} be another. Now we've reduced $m+1$ sets to 2 sets. Then we can simply apply De Morgan's law for $n = 2$ to show $\varphi(m+1)$ holds. Hence $\varphi(k)$ holds for all $k \geqslant 1$. QED.

Problem 3 (1.6)**.** Why is the square of an odd integer odd and the square of an even integer even? What is the situation for higher powers?

Solution. If n is odd, then $\exists k \in \mathbb{Z}$ such that $n = 2k+1$. It follows that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2+k)+1$, another odd number. If n is even, then $\exists k \in \mathbb{Z}$ such that $n = 2k$. Then $n^2 = (2k)^2 = 4k^2$, even.

If x is odd, then its prime factorization contains no 2. Since the prime factorization of x^n simply multiplies each exponent in x's prime factorization by n, there is still no 2. Therefore x^n is still odd. On the other hand, if x is even, then it has 2 in its prime factorization. Therefore x^n also has 2 in its prime factorization, and it's even.

(I could have said the same thing to square numbers, but Alas $(2k+1)$ makes everything look fancier.)

Problem 4 (1.8)**.** Suppose that the natural number k is not a perfect n^{th} power.

(1) Prove that its n^{th} root is irrational.

(2) Infer that the n^{th} root of a natural number is either a natural number or it is irratitonal. It is never a fraction.

Solution.

(1) First thing: $k = 1$ is trivial, so we will focus on $k \geqslant 2$. Suppose the n^{th} root of k is rational, then it can be written as $\dfrac{p}{q}$ where p and q are co-prime integers with $q \neq 0$. It follows that $\left(\dfrac{p}{q}\right)^n = k$ and thus $p^n = kq^n$.

By prime factorization, p can be written as $\prod p_i^{\ell_i}$, a product of primes, each raised to some powers ($\geqslant 1$). Clearly the LHS is a multiple of p_1, and the RHS must also be. Two possibilities:

I. p_1 divides q^n. If this is the case then it means p_1 appears in the prime factoziation of q^n. Since the prime factorizations of q^n and q use the same set of prime bases, p_1's appearance in the prime factorization of q^n implies its appearance in the prime factorization of q, i.e., p_1 divides q. However, we have assumed at the first place that p and q are co-prime. Contradiction.

II. If p_1 doesn't divide q^n, then it must divide k. We actually need a stronger argument here. Since $p_1^{\ell_1}$ is a divisor of p, the statement still holds when we raise both to the n^{th} power, i.e., $p_1^{n\ell_1}$ divides p^n. Therefore, $p_1^{n\ell_1}$ must also divide kq^n. We claimed that even p_1 doesn't divide q^n, so the entire $p_1^{n\ell_1}$ must divide k. For now we will leave it like this.

Note that, when analyzing the divisibility of p_1, we immediately run into contradiction if it's case (I) but if we get case (II) everything seems fine. Like previously said, if we ended up running into case (II), we will temporarily skip it and start analyzing the divisibility of p_2 analagously. Again, we either run into the contradiction that p and q are not co-prime, or we run into a similar case (II) for p_2, in which case we will look at p_3 and so on...

If we ended up getting case (I) for any p_i, we immediately terminate the divisibility check and conclude that the n^{th} root of k is irrational since p and q are not co-prime and this contradicts our assumption. If we managed to "survive" all the way till we are done the last p_i, then indeed p and q are co-prime. But now look at k. It has factors $p_1^{n\ell_1}, p_2^{n\ell_2}, \ldots$ for every single p_i that divides p, and it does not have any other prime factors outside the list of p_i's because the LHS, p^n, is not divisible by other primes. Therefore, since in the prime factorization of k each prime factor is raised to the power of a multiple of n, we have $k = (\prod p_i^{\ell_i})^n$, a perfect n^{th} power itself. Contradiction... finally.

Having considered all possibilities, we may finally conclude that if k is not a perfect n^{th} power, its n^{th} root is irrational. □

(2) Suppose some fraction $\frac{p}{q}$ satisfies $\left(\frac{p}{q}\right)^n = k$ where k isn't a perfect n^{th} power and p, q are co-prime with $q \neq 0$. (We have set k to not be a perfect n^{th} power because otherwise its n^{th} root is simply a natural number, not the fraction we are looking for.) Then $p^n = kq^n$, and by part (1) we know that some contradiction will come up, either p and q are not co-prime or k is a perfect n^{th} power. Therefore there does not exist a fraction that can become an integer when raised to some power.

† Linfeng helped me realize the necessity of resorting to prime factorization to prevent a potential loophole in the question above. Credits & kudos to him / you (I know you are reading this).

Problem 5. Show that in general $(A \setminus B) \cup B \neq A$.

Solution. Check out the Venn diagram in problem 2 again. $A \setminus B$ refers to the cyan region, and $(A \setminus B) \cup B$ refers to $A \cup B$, which is different from A unless $B \setminus A = \varnothing$, i.e., $B \subseteq A$.

Problem 6. Given an example of a binary relation which is

(1) reflexive and symmetric, but not transitive;

(2) reflexive, but neither symmetric nor transitive;

(3) symmetric, but neither reflexive nor transitive; and

(4) transitive, but neither reflexive nor symmetric.

Solution. Consider the set $\mathcal{S} = \{1, 2, 3\}$. We will come up with four relations.

(1) Define relation R_1 as $\{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$. R_1 is reflexive because $(1,1), (2,2)$, and $(3,3)$ are all in R_1. It is symmetric because if $(a,b) \in R_1$ then we can also find (b,a) in R_1. It is, however, not transitive: $1R_12 \wedge 2R_13$ does not give us $1R_13$.

(2) Define relation R_2 as $\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$. R_1 is reflexive because $(1,1),(2,2)$, and $(3,3)$ are all in R_2. R_2 is not symmetric because $1R_22$ does not lead to $2R_21$, and it's not reflexive because $1R_22 \wedge 2R_23$ do not lead to $1R_23$.

(3) Define relation R_3 as $\{(1,2),(2,1),(2,3),(3,2)\}$. It is not reflexive becuase $(1,1) \notin R_3$. It is not transitive because $1R_32 \wedge 2R_33$ do not lead to $1R_33$. However it is indeed symmetric: $aR_3b \implies bR_3a$ for all $(a,b) \in R_3$.

(4) Define relation R_4 as $\{(1,2),(2,3),(1,3)\}$. It is not reflexive because $(1,1) \notin R_4$. It is not symmetric because $1R_42$ but not $2R_41$. It is, however, transitive: the only pair $(a,b),(b,c) \in R_4$ we can find is $(1,2),(2,3)$, and indeed we have $1R_43$.