MATH 425a Problem Set 10

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Problem 1

Consider a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable and satisfies $\lim_{x\to\infty} f'(x) = 0$. Prove that we have $\lim_{x\to\infty} (f(x+1) - f(x)) = 0$.

Solution

By MVT, given x, there exists some $y \in (x, x+1)$ such that

$$f'(y) = \frac{f(x+1) - f(x)}{(x+1) - x} = f(x+1) - f(x).$$

Since $\lim_{x\to\infty} f'(y) = \lim_{x\to\infty} f'(x) = 0$, so does $f(x+1) - f(x)$ as $x \to \infty$.

Problem 2

Let $q_1, q_2, \dots$ be an enumeration of the set of rational numbers in $(0,1)$. Define a function $f : (0,1) \to \mathbb{R}$ by

$$f(x) = \sum_{q_n < x} 2^{-n}.$$

Prove that f is continuous at every irrational number and discontinuous at every rational number.

Solution

We will first look at the irrational case. Pick any irrational $x \in (0,1)$ and $\epsilon > 0$. Then, there exists $n \in \mathbb{N}$ such that $1/2^n < \epsilon$.

We first show that there exists $\delta > 0$ such that if $y \in (x, x+\delta) \implies 0 < f(y) - f(x) < \epsilon$. Among the first n

terms of the enumeration, there exists a q_i such that it's the smallest rational number greater than x. Let the difference be δ_1. Then all rational in the interval $(x, x+\delta_1)$ appear no earlier than q_{n+1} in the enumeration. Therefore, if $y \in (x, x+\delta_1)$ then

$$f(y) - f(x) = \sum_{q_i \in [x, x+\delta_1]} 2^{-i} \leqslant \sum_{q_i \in (x, x+\delta_1)} 2^{-i} \leqslant \sum_{q_i | i \geqslant n+1}^{\infty} 2^{-i} = \frac{1}{2^n} < \epsilon.$$

Now consider the largest rational less than x among the first n terms of the enumeration. Let the difference between it and x be δ_2. Again, all rationals in $(x-\delta_2, x)$ appear no earlier than q_{n+1} in the enumeration. Hence if we pick $y \in (x-\delta_2, x)$, we get

$$f(x) - f(y) = \sum_{q_i \in (x-\delta_2, x)} 2^{-i} \leqslant \sum_{q_i | i \geqslant n+1}^{\infty} 2^{-i} = \frac{1}{2^n} < \epsilon.$$

Then, taking $\delta \coloneqq \min(\delta_1, \delta_2)$ suggests f meets the $\epsilon - \delta$ condition at irrationals and is therefore continuous at irrationals.

If x is rational in $(0,1)$, it is of form q_n. For any $y > x$ we have $f(y) - f(x) \geqslant 2^n$ since $x \nless x$ but $x < y$. Letting $\epsilon < 0$ we see no $\delta > 0$ guarantees the $\epsilon - \delta$ continuity at x. Hence f is not continuous at the rationals.

Problem 3: 3.1 (Pugh)

Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leqslant |t-x|^2$ for all t, x. Prove that f is constant.

Solution

This inequality gives rise to the fact that

$$-|t-x| \leqslant -\frac{|f(t)-f(x)|}{|t-x|} \leqslant \frac{f(t)-f(x)}{t-x} \leqslant \frac{|f(t)-f(x)|}{|t-x|} \leqslant |t-x|.$$

Therefore, taking the limits as $t \to x$ suggests

$$0 = \lim_{t\to x}(-|t-x|) \leqslant \lim_{t \to x} \frac{f(t)-f(x)}{t-x} = f'(x) \leqslant \lim_{t\to x} |t-x| = 0.$$

Therefore $f'(x) = 0$ for all x, i.e., f is constant.

Problem 4: 3.3 (Pugh)

Assume $f : (a,b) \to \mathbb{R}$ is differentiable.

(1) If $f'(x) > 0$ for all x, prove that f is strictly monotone increasing.

(2) If $f'(x) \geqslant 0$ for all x, what can you prove?

Solution

(1) Suppose f is not strictly increasing. Then there exist $x_1 < x_2$ such that $f(x_1) \geqslant f(x_2)$. Therefore, by mean value theorem, there must exist some $x_3 \in (x_1, x_2)$ satisfying

$$f'(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

which would be either 0 or < 0, contradicting the assumption that $f'(x)$ is always positive. Hence the function is strictly monotone increasing, i.e., $x_2 > x_1 \implies f(x_2) > f(x_1)$.

(2) This would mean that f is still monotone increasing, albeit not strictly, as $x_2 > x_1 \implies f(x_2) \geqslant f(x_1)$.

Problem 5: 3.4 (Pugh)

Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

Solution

Note that

$$\begin{aligned}
0 < \sqrt{n+1} - \sqrt{n} &= (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&< \frac{1}{2\sqrt{n}}.
\end{aligned}$$

Since $2\sqrt{n} \to \infty$ as $n \to \infty$ we have

$$0 \leqslant \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} \leqslant \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0.$$

Therefore $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

Problem 6: 3.8(a) (Pugh)

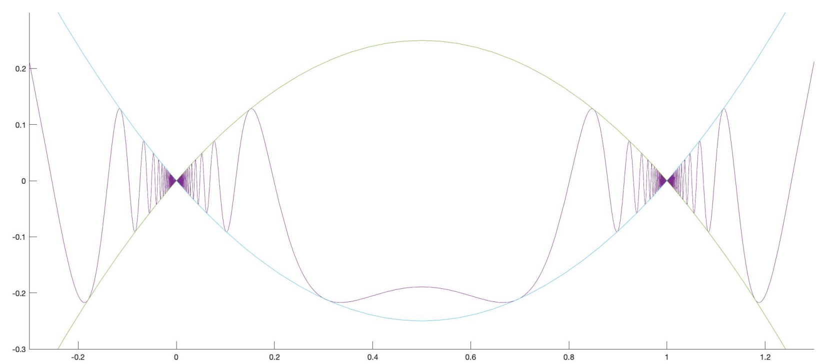
Draw the graph of a continuous function defined on $[0,1]$ that is differentiable on the interval $(0,1)$ but not at the endpoints.

Solution

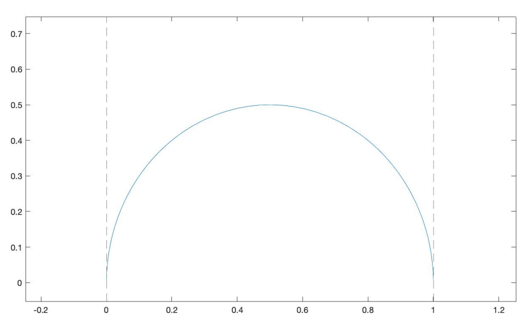
I was thinking about $f(x) = g(x)\sin(1/g(x))$ such that $g(x) \to 0$ as $x \to 0$ and $x \to 1$. Then I came up with $g(x) = x(1-x)$ and this function:

$$f(x) = \begin{cases} 0 & x = 0, 1 \\ g(x) & \text{otherwise} \end{cases}$$

whose graph looks like the following.



But then Bruno suggested something *much, much* simpler: $f(x) = \sqrt{0.25 - (x-0.5)^2}$.



Problem 7: 3.9 (Pugh)

Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable.

(1) If there is an $L < 1$ such that for each $x \in \mathbb{R}$ we have $f'(x) < L$, prove that there exists a unique point x such that $f(x) = x$.

(2) Show by example that (1) fails if $L = 1$.

Solution

(1) Define $g(x) := f(x) - x$ so that $g'(x) = f'(x) - 1 < L - 1 < 0$ for all x. Since

$$g(x) = g(0) + \int_0^x g'(\tilde{x})\, \mathrm{d}\tilde{x} < g(0) + (L-1)x$$

we see that $g(x_1) < 0$ for sufficiently large x_1 and $g(x_2) > 0$ for sufficiently small (maybe negative) x_2. Since f is differentiable it is continuous, and so is g. Therefore $g(c) = 0$ for some $c \in (x_1, x_2)$, and at this point we know $f(c) = c$, i.e., c is a fixed point of f.

(2) Consider the Signoid function $1/(1 + e^{-x})$. If we define $g(x)$ as

$$g(x) := 1 - \frac{1}{1 + e^{-x}}$$

then we see that the range of $g(x)$ is $(1, 0)$. Taking its antiderivative gives

$$G(x) = \int 1 - \frac{1}{1 + e^{-x}}\, \mathrm{d}x = x - \ln(e^x + 1) + C.$$

Setting $C = 0$, we get a function with derivative < 1 without a fixed point, as $\ln(e^x + 1) \neq 0$ for all $x \in \mathbb{R}$.

Problem 8: (extra credit) 1.31 (Pugh)

Suppose that a function $f : [a, b] \to \mathbb{R}$ is monotone nondecreasing, i.e., $x_1 \leqslant x_2 \implies f(x_1) \leqslant f(x_2)$.

(1) Prove that f is continuous except at a countable set of points.

(2) Is the same assertion true for a monotone function defined on all of $\mathbb{R}$?

Solution

We'll first prove the following lemma. Once done, the rest is almost immediate.

Lemma

Suppose $f : [a, b] \to \mathbb{R}$ is monotone nondecreasing. Then f is continuous at x if and only if the jump at x is zero, i.e., $f(x+) = f(x-)$.

Proof

Since f is monotone nondecreasing, we can define

$$f(x+) := \inf\{f(\tilde{x}) \mid \tilde{x} > x\} \text{ and } f(x-) := \sup\{f(\tilde{x}) \mid \tilde{x} < x\}.$$

For $\Longrightarrow$, suppose f is continuous at x. Then given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}.$$

By the monotonicity of f, it follows that $f(x+) = \inf\{f(\tilde{x}) \mid x \in (x, x + \delta)\} < f(x) + \epsilon/2$. Likewise $f(x-) < f(x) - \epsilon/2$. Therefore $f(x+) - f(x-) < \epsilon$. Since ϵ is arbitrary, we conclude that f has zero jump at x.

Now for the converse $\Longleftarrow$, suppose $f(x+) = f(x-) = a$. By the properties of infimum and supremum, given $\epsilon > 0$, there exist $b > x$ with $f(b) < a+\epsilon$ and $c < x$ with $f(c) > a-\epsilon$. If we define $\delta := \min\{(b-x), (x-a)\}$ then

$$|x - y| < \delta \implies |a - f(y)| < \epsilon. \tag{*}$$

Here the problem reduces to showing that $f(x) = a$. Suppose not and WLOG assume $f(x) < a$. Then there exists some $t > x$ such that $f(t) \in (f(x), a) \implies f(t) < f(x)$. This contradicts f's being monotone nondecreasing. Likewise for the other case. Hence $f(x) = a$, and the above $(*)$ becomes

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon,$$

i.e., f is continuous at x. □

(1) Once more, by the monotonicity of f, if x_1, x_2 are two points of discontinuity, then

$$f(x_1-) \leqslant f(x_1) \leqslant f(x_1+) \leqslant f(x_2-) \leqslant f(x_2) \leqslant f(x_2+) \text{ with } f(x_1) < f(x_2).$$

Therefore, there exists a bijection between the set $\mathcal{S}$ of all points of discontinuity and a bunch of open intervals of form $(f(x-), f(x+))$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists (at least) one rational in each interval, and so there exists an injection $g : \mathcal{S} \to \mathbb{Q}$. This implies $\mathcal{S}$ is countable.

(2) Yes. By (1), for any $n \in \mathbb{Z}$, there exist only countably many points of discontinuity in the interval $[n, n+1]$ and thus in $[n, n+1)$. The union of all these intervals is precisely $\mathbb{R}$. Hence the same assertion still holds.