MATH 425a Problem Set 11

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Problem 1

Prove that a countable union of null sets in $\mathbb{R}$ is again a null set. What about an uncountable union?

Solution

Let $\{\mathcal{S}_i\}$ be an enumeration of these null sets. It follows that, given $\epsilon > 0$, we can cover $\mathcal{S}_1$ with countably many open intervals with total length $< \epsilon/2$. Likewise, we can cover $\mathcal{S}_i$ with countably many open intervals with total length $\epsilon/2^i$. Therefore we can their union with countably many open intervals of total length $< \epsilon/2 + \epsilon/4 + \cdots = \epsilon$. Hence the union is again a null set.

The assertion fails for an uncountable union: on one hand singles are null sets since $\{x\}$ can be covered by $(x-\epsilon, x+\epsilon)$ for any $\epsilon > 0$, while on the other hand

$$[0,1] = \bigcup_{i\in[0,1]} \{i\}$$

is not a null set. The proof will be given in the extra credit problem later on.

Problem 2

Let $f : [0,1] \to \mathbb{R}_{>0}$ be a continuous function. Prove that we have $\displaystyle\int_0^1 f(x)\,\mathrm{d}x > 0$. What happens if we assume f is Riemann integrable but not necessarily continuous?

Solution

Claim: if f is Riemann integrable then the integral still needs to be strictly positive. Since if $f : [0,1] \to \mathbb{R}_{>0}$ is continuous it's automatically Riemann integrable, it suffices to simply prove the second question.

First notice that, since f is always positive, it's impossible that its Riemann integral evaluates to a negative value since $\int_0^1 f(x)\,\mathrm{d}x = \overline{I} = \lim U(f,P)$ whereas each upper sum $U(f,P)$ is positive. Now assume, for contradiction, that $\int_0^1 f(x)\,\mathrm{d}x = 0$. It follows that there exists a partition P_1 of $[0,1]$ such that $U_{[0,1]}(f,P_1) < 1$. Clearly, under P_1, some subinterval $[x_1,y_1]$ has to satisfy

$$\sup_{x\in[x_1,y_1]} f(x) < 1 \text{ or otherwise } U_{[0,1]}(f,P_1) \geqslant 1.$$

Now we narrow down our focus to $\int_{x_1}^{y_1} f(x)\,\mathrm{d}x = 0$. Similarly there has to exist a partition P_2 such that $U_{[x_1,y_1]}(f,P_2) < (y_1 - x_1)/2$, and so there exists some subinterval $[x_2,y_2]$ from P_2 such that

$$\sup_{x\in[x_2,y_2]} f(x) < \frac{1}{2} \text{ or otherwise } U_{[x_1,y_1]}(f,P_2) \geqslant \frac{y_1 - x_1}{2}.$$

We may continue doing so and get a nested sequence of intervals $([x_i,y_i])$. By Cantor Intersection Theorem,

$$[x,y] := \bigcap_{i=1}^{\infty} [x_i,y_i] \text{ is compact and nonempty,}$$

so if we pick $x_0 \in [x,y]$ we have $f(x_0) < 1/n$ for all $n \in \mathbb{N}$ so $f(x_0) \leqslant 0$. But this contradicts the assumption that $f : [0,1] \to \mathbb{R}_{>0}$. Hence $\int_0^1 f(x)\,\mathrm{d}x > 0$.

Problem 3: 3.14 (Pugh)

For each $r \geqslant 1$, find a function that is C^r but not C^{r+1}.

Solution

Similar to the examples provided in the textbook, consider the following:

$$f(x) = \begin{cases} x^{r+1} & x \geqslant 0 \\ -x^{r+1} & x < 0 \end{cases}$$

Problem 4: 3.27 (Pugh)

In many calculus books, the definition of the integral is given as

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\frac{b-a}{n}$$

where x_k^* is the midpoint of the k^{th} interval of $[a,b]$ having length $(b-a)/n$, namely

$$[a+(k-1)(b-a)/n, a+k(b-a)/n].$$

See Stewart's *Calculus: Early Transcendentals*, for example.

(1) If f is continuous, show that the calculus-style limit exists and equals the Riemann integral of f.

(2) Show by example that the calculus-style limit can exist for functions which are not Riemann integrable.

(3) Infer that the calculus-style definition of the integral is inadequate for real analysis.

Solution

(1) Since $[a,b]$ is compact and f is continuous we know f is also uniformly continuous. Given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x)-f(y)| < \frac{\epsilon}{b-a}.$$

If we pick n large enough such that $(b-a)/n < 2\delta$, then in each subinterval, the distance between the midpoint and any other point is $< \delta$. Fix these endpoints and define them as partition P of $[a,b]$. It follows that

$$\left|\sum_{k=1}^{n} f(x_k^*)\frac{b-a}{n} - R(f,P,T)\right| \leqslant \sum_{k=1}^{n}\left|(f(x_k^*)-f(t))\frac{b-a}{n}\right| < n\cdot\frac{\epsilon}{b-a}\cdot\frac{b-a}{n} = \epsilon.$$

Hence the two styles are equivalent for a continuous function.

(2) Consider Dirichlet's function restricted on $[0,1]$:

$$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

The midpoint of each subinterval is rational, so the calculus-style gives $\int_0^1 f(x)\,\mathrm{d}x = 1$. However, since

$\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$, given any nondegenerate interval $[a, b]$, we always have

$$\inf_{x \in [a,b]} f(x) = 0 \text{ and } \sup_{x \in [a,b]} f(x) = 1,$$

and thus f is not Riemann integrable.

(3) See example above. We need a better way (*Riemann-Lebesgue Theorem*) to determine whether a function is Riemann integrable or not.

Problem 5: 3.32 (Pugh)

Consider the characteristic function of the dyadic rational numbers:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a dyadic rational} \\ 0 & \text{otherwise} \end{cases}$$

(1) What is its set of discontinuities?

(2) At which points is its oscillation $\geqslant \epsilon$?

(3) Is it (Riemann) integrable? Explain, both by the Riemann-Lebesgue Theorem and directly from the definition.

(4) Consider the **dyadic ruler function**. Graph it and answer the above questions.

$$g(x) := \begin{cases} 1/2^n & \text{if } x \text{ is a dyadic rational} \\ 0 & \text{otherwise} \end{cases}$$

Solution

(1) This function is discontinuous everywhere. Let $\epsilon = 1/2$. Pick any $x \in \mathbb{R}$. No matter how small $\delta > 0$ is, there will always be dyadic rationals of form $k/2^n$ between $(x - \delta, x + \delta)$ where n is large enough that $1/2^n < \delta$. Clearly since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, there will also be irraitonals (non-dyadic) in the interval. Hence

$$\sup_{y \in (x-\delta, x+\delta)} |f(x) - f(y)| = 1 > \epsilon \implies f \text{ is not continuous at } x.$$

(2) Everywhere for $\epsilon \leqslant 1$ and nowhere for $\epsilon > 1$.

(3) It is not Riemann integrable. First approach: regardless of the size of mesh, all nondegenerate intervals

$[a,b]$ satisfy

$$\inf_{x\in[a,b]} f(x) = 0 \text{ and } \sup_{x\in[a,b]} f(x) = 1.$$

Therefore it's impossible for the lower and upper sums to agree, hence f is not Riemann integrable.

Alternatively, from (1) we see that the function is discontinuous everywhere. Since the set of discontinuity points is not a null set, by Riemann-Lebesgue Theorem f is not Riemann integrable.

(4) Unlike the previous question, the dyadic ruler function is discontinuous only at dyadic rationals. Also notice that this function is periodic with period 1. Therefore it suffices to check its behavior on $[0,1]$.

Pick any dyadic rational $q = k/2^n$. Should f be continuous, if we set $\epsilon := k/2^{n+1}$ we should be able to find a $\delta > 0$ such that $x \in (q-\delta, q+\delta) \implies |f(x) - f(q)| < \epsilon$. This means $f(x) \in (k/2^{n+1}, 3k/2^{n+1})$, but we know this is impossible because the non-dyadic numbers are dense in $\mathbb{R}$ (take the irrationals for example). Each interval $(q-\delta, q+\delta)$ contains non-dyadic numbers that get mapped to 0 whereas $0 \notin (k/2^{n+1}, 3k/2^{n+1})$. Therefore g is discontinuous at dyadic rationals.

On the other hand, g is continuous at non-dyatic numbers. To see this, pick $p \in \mathbb{R}$ that's not a dyadic rational so $g(p) = 0$. Given $\epsilon > 0$, there exists a $n \in \mathbb{N}$ such that $1/2^{n+1} < \epsilon < 1/2^n$. Let

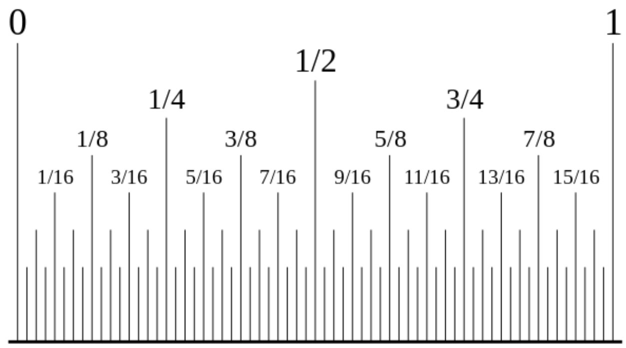
$$\mathcal{P} = \{p_0, p_1, \ldots, p_{2^n}\} := \left\{0, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{2^n - 1}{2^n}, 1\right\}$$

be a partition of $[0,1]$. It follows that if we define

$$\delta := \inf_{p_i \in \mathcal{P}} |p - p_i|$$

then the open interval $(p-\delta, p+\delta)$ does not contain any dyadic rationals whose denominator $\leqslant 2^n$. Therefore anything in $(p+\delta, p-\delta)$ is either a non-dyadic number that evaluates to 0 or a dyadic rational with denominator $\geqslant 2^{n+1}$ which evaluates to $\leqslant 1/2^{n+1} < \epsilon$. From this we see that the $(\epsilon - \delta)$ condition is met. Hence g is continuous at non-dyadic numbers.

Since the set of dyadic numbers is a subset of $\mathbb{Q}$, it is countable and therefore a null set. By Riemann-Lebesgue Theorem, g is Riemann integrable.



Graph of dyadic ruler function on $[0,1]$. Source: *Wikipedia.*

Problem 6: 3.53 (Pugh)

Given $f, g \in \mathcal{R}$, prove that the pointwise $\max(f,g)$ and $\min(f,g)$ are Riemann integrable.

Solution

By the Riemann integrability of f and g, given $\epsilon > 0$ there exist partitions P_1, P_2 of $[a,b]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2} \text{ and } U(g,P_2) - L(g,P_2) < \frac{\epsilon}{2}.$$

Define $T := P_1 \cup P_2$, a "refined" partition of $[a,b]$. It follows that $U(f,T) - L(f,T) \leqslant U(f,P_1) - L(f,P_1)$ and likewise for g.

Now we show that $\max(f,g) \in \mathcal{R}$. Notice that, on each interval $[a,b]$,

$$\sup_{x\in[a,b]} \max(f,g)(x) = \max\{\sup_{x\in[a,b]} f(x), \sup_{x\in[a,b]} g(x)\}$$

and

$$\inf_{x\in[a,b]} \max(f,g)(x) \geqslant \max\{\inf_{x\in[a,b]} f(x), \inf_{x\in[a,b]} g(x)\}$$

($\geqslant$ because the actual infimum of f and g may or may not be attained). Abbreviating some notations gives

$$\begin{aligned} \sup\max(f,g) - \inf\max(f,g) &\leqslant \max\{\sup f(x) - \inf f(x), \sup g(x) - \inf g(x)\} && \text{(simple case check)} \\ &\leqslant \sup f(x) - \inf f(x) + \sup g(x) - \inf g(x). && \text{(sup } - \text{ inf is nonnegative)} \end{aligned}$$

Therefore,

$$U(\max(f,g),T) - L(\max(f,g),T) \leqslant U(f,T) - L(f,T) + U(g,T) - L(g,T) < \epsilon,$$

so $\max(f,g)$ is Riemann integrable. The proof for $\min(f,g)$ is analogous using the $\epsilon/2$ argument — all we need to do is to swap inf and sup as well as $\geqslant$ and $\leqslant$.

Problem 7: (extra credit) 3.29 (Pugh)

Prove that the interval $[a,b]$ is not a null set. Also explain why the following observation is not a solution to the problem: "*every open interval that contains* $[a,b]$ *has length* $> b-a$."

Solution

That argument fails because it does not eliminate the possibility of finding a countable open covering of $[a,b]$ using smaller intervals whose total lengths $< b-a$.

Now suppose we had a "bad" covering $\{I_i\}$ of $[a,b]$ whose total interval length $< b-a$. By the compactness of $[a,b]$ we may reduce $\{I_i\}$ to finite subcovering. Define $\mathcal{B} := \{I_1, I_2, \ldots, I_n\}$ to be the minimal bad covering such that no $n-1$ open intervals can cover $[a,b]$ with total length $< b-a$. Clearly $n \neq 1$ or we would have one long interval of total length $> b-a$. Otherwise, since a is covered by $\mathcal{B}$, WLOG $a \in I_1$. Then I_1 has form (x_1, y_1) where $x_1 < a < y_1 < b$. It follows that $y \in I_2$ for some other interval which we call $I_2 = (x_2, y_2)$ or otherwise $y \in [a,b]$ is not covered by $\mathcal{B}$. It follows that $x_2 < y_1$ so $I_1 \cap I_2 \neq \varnothing$. Then $I := I_1 \cup I_2 = (x_1, y_2)$, an open interval, and $|I| = |I_1| + |I_2| - |I_1 \cap I_2| < |I_1| + |I_2|$. Now we've got a new covering $\mathcal{B}' := \{I, I_3, \ldots, I_n\}$ of $n-1$ scraps of even less total length that manages to cover $[a,b]$. Contradiction. Hence no open covering of $[a,b]$ can have total length $< b-a$. □