MATH 425a: Problem Set 2

Qilin Ye

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Problem 1 (1.9)**.** Let $x = A \mid B, x' = A' \mid B'$ be cuts in $\mathbb{Q}$. We defined

$$x + x' = (A + A') \mid \text{the rest of } \mathbb{Q}.$$

(1) Show that although $B + B'$ is disjoint from $A + A'$, it may happen in degenerate cases that $\mathbb{Q}$ is not the union of $A + A'$ and $B + B'$.

(2) Infer that the definition of $x + x'$ as $(A + A') \mid (B + B')$ would be incorrect.

(3) Why did we not define $x \cdot x' = (A \cdot A') \mid \text{rest of } \mathbb{Q}$?

Solution.

(1) It turns out that sometimes $(A + A') \cup (B + B')$ would be missing an element of $\mathbb{Q}$. Before we begin, define set $\mathcal{S} = \{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 < 2\}$, namely all positive rational numbers less than $\sqrt{2}$ (note that we cannot actually use this definition because $\sqrt{2}$ is not in $\mathbb{Q}$.) Now consider sets

$$x = A \mid B = \{r \in \mathbb{Q} \mid r \leqslant 0 \text{ or } r \in \mathcal{S}\} \mid \text{the rest of } \mathbb{Q}$$
$$x' = A' \mid B' = \{r \in \mathbb{Q} \mid r \leqslant 0 \text{ or } r < 2 - s, \forall s \in \mathcal{S}\} \mid \text{the rest of } \mathbb{Q}$$

Though I want to define the set A' as the set of all rational numbers less than $2 - \sqrt{2}$, but again we cannot say so because, again, $2 - \sqrt{2}$ is not a rational number.

Having constructed the two sets without involving $\mathbb{R}$, hopefully now I may use $\sqrt{2}$ now. Since $a < \sqrt{2}$ for all $a \in A$ and $a' < 2 - \sqrt{2}$ for all $a' \in A'$, it follows that all elements of $A' + B'$ are strictly less than $\sqrt{2} + (2 - \sqrt{2}) = 2$. Therefore $2 \notin A + A'$.

On the other hand, since $\sqrt{2}$ is irrational, $B = \{r \in \mathbb{Q} \mid r > 0 \text{ and } r^2 \geqslant 2\}$ is the same as $\{r \in \mathbb{Q} \mid r > 0 \text{ and } r^2 > 2\}$. In other words, all elements of B are strictly larger than $\sqrt{2}$. Likewise, all element of B' are strictly larger than $2 - \sqrt{2}$. Therefore, all elements of $B + B'$ are strictly larger than $\sqrt{2} + (2 - \sqrt{2}) = 2$. Hence $2 \notin B + B'$ either. Therefore $(A + A') \cup (B + B')$ is not equal to $\mathbb{Q}$.

(2) Based on the definion, if $X \mid Y$ is a Dedekind cut then $X \cup Y = \mathbb{Q}$ which is not the case for $(A + A')$ and $(B + B')$ as shown in (1).

(3) We do not define multiplication as such because the product of two negative numbers is positive, and so $(A \cdot A')$ can give us arbitrarily large positive numbers which we do not want.

Problem 2 (1.10)**.** Prove that for each cut x we have $x + (-x) = 0^*$. **This is not trivial.**

Solution. The proof involves the assumption that $\mathbb{Q}$ has the Archimedean property. For clarity, we will denote $0^* = A_0 \mid B_0$, $x = A \mid B$, and $-x = A' \mid B'$. The first case is when $x = -x = 0^*$. In this case $A_0 = A = A'$ and the proposition $x + (-x) = 0^*$ is trivial.

Now, if $x, (-x) \neq 0$, without loss of generality (WLOG) we may assume $x > 0$, i.e., $A_0 \subsetneq A$. To show $x + (-x) = 0^*$ or $A + A' = A_0$, we need to show both $(A + A') \subset A_0$ and $(A + A') \supset A_0$.

Showing $\subset$ is easy. Since $\forall b \in B$ and $\forall a \in A$ we always have $b > a$, it follows that, for all $a \in A$ and for all $b \in B$ such that $\exists c > 0$ and $-b - c \notin A$ (i.e., for all elements of B excluding the smallest one — recall how A' of $(-x)$ is defined), $a + (-b) < 0$ always holds, i.e., $A + A' \subset A_0$.

Now for the $\supset$ part. First pick an arbitrary $m = A_m \mid B_m$ satisfying $A_m \subsetneq A_0$, i.e., $m < 0$. Clearly m is negative. Now set $n = -\frac{m}{2}$, clearly a positive number.[1] Since $\mathbb{Q}$ has the Archimedean property, there exists an integer k such that

$$\begin{cases} kn \in A, & \text{i.e.,} \quad kn < x \\ (k+1)n \in B, & \text{i.e.,} \ (k+1)n \geqslant x \end{cases}$$

Now let $j = -(k+2)n$. Recall the definition of $-x$ when $x = A \mid B$:

$$-x = C \mid D = \{r \in \mathbb{Q} \mid \text{ for some } b \in B, \text{ not the smallest element of B}, r = -b\} \mid \text{the rest of } \mathbb{Q}$$

If we look at $-j = (k+2)n$, we know $-j > (k+1)n$ since n is positive. Therefore $-j$ is not the smallest element of B and thus $j \in A'$. Now we have $kn \in A$ and $j \in A'$. Adding them together gives

$$kn + j = kn - (k+2)n = -2n = m.$$

Therefore any arbitrary $a_0 \in A_0$ can be written as the sum of some $a \in A$ and $a' \in A'$. We have $(A + A') \supset A_0$.

Having shown $(A + A') \subset A_0$ and $(A + A') \supset A_0$, we conclude that $(A + A') = A_0$ and thus $x + (-x) = 0^*$. Really not trivial... □

Problem 3 (1.12)**.** Prove that there exists no smallest positive real number. Does there exist a smallest positive rational number? Give a real number x, does there exist a smallest real number $y > x$?

Solution. All answers are NO.

[1] Rudin W. *Principles of Analysis.* 3^{rd} ed., McGraw-Hill, 1976, p.19.

(1) Suppose $x > 0$ is the smallest positive real number. Since we know $1 > \frac{1}{2}$, it follows that $1 \cdot x > \frac{1}{2} \cdot x \implies x > \frac{x}{2}$, and we've just found a smaller positive real number. Contradiction. Therefore there does not exist a smallest positive real number.

(2) Similar to (1) except the word "real" is replaced by "rational".

(3) Suppose no, then the statement implies there is no real number x such that $x > 0$ and $x < y - x$. Otherwise we can simply add the smaller number to x and find a real number greater than x but smaller than y. By (1) we are always able to come up with such numbers. Therefore, given x, there does not exist a smallest real number $y > x$.

Problem 4 (1.14)**.** Prove that $\sqrt{2} \in \mathbb{R}$ by showing that $x \cdot x = 2$ where $x = A \mid B$ is the cut in $\mathbb{Q}$ with $A = \{r \in \mathbb{Q} \mid r \geqslant 0 \text{ or } r^2 < 2\}$.

Solution. Recall that the definition of $x \cdot x$ where $x = A \mid B$ is

$$x \cdot x = C \mid D = \{r \in \mathbb{Q} \mid r \leqslant 0 \text{ or } \exists \text{ positive } a, b \in A \text{ such that } r = ab\} \mid \text{the rest of } \mathbb{Q}.$$

Also note that $2 = E \mid F = \{x \in \mathbb{Q} \mid x < 2\} \mid \{x \in \mathbb{Q} \mid x \geqslant 2\}$. To show $x \cdot x = 2$, we have to show $(C \subset E) \wedge (C \supset E)$ so that $C = E$.

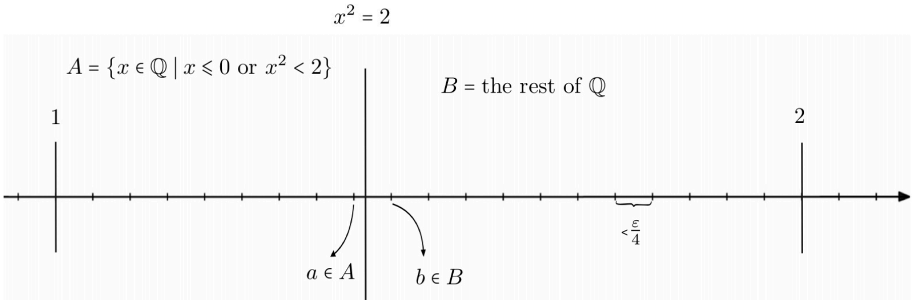
Like the first problem, showing $\subset$ is super easy: if a, b satisfy $a^2, b^2 < 2$, then $ab < 2$. Therefore $C \subset E$.

On the other hand, we want to show $E \supset F$. A formal way to phrase this statement is the following:

$$\boxed{\text{For each } \epsilon > 0, \text{ there exists } a \in A \text{ such that } 2 - \epsilon < a^2 < 2.}$$

In other words, if $2 - \epsilon = c \in C$ then there exists $a^2 \in E$ such that $a^2 > c$ so $c \in E$ as well.

Now imagine $1, 2$ sitting on an evenly spaced axis of rational numbers. We may divide the interval between 1 and 2 into many sub-intervals, each with a rational length $< \epsilon/4$. (We can do so by simply picking an integer $> 4/\epsilon$ and set the interval length to $1/n$.) Again, by the Archimedean property of $\mathbb{Q}$, among all the endpoints of these sub-intervals, there exists an adjacent pair, where the square of the value of the left one < 2 and the other > 2. To visualize this, see the not-to-scale diagram below (on the next page).



Since $a < 2$ and $b < 2$, we know $a + b < 4$. Moreover, we also know $b - a = \epsilon/4$. Therefore $b^2 - a^2 = (b+a)(b-a) < \epsilon$. Since $b^2 > 2$ and $a^2 < 2$, we know that the point 2 is between a and b. Therefore $2 - a^2 < b^2 - a^2 < \epsilon$, and we've successfully found an a such that $2 - \epsilon < a^2 < 2$. Hence $C \supset E$ and therefore $C = E$, $x \cdot x = 2$. □

Problem 5 (1.15)**.** Given $y \in \mathbb{R}$, $n \in \mathbb{N}$, and $\epsilon > 0$, show that for some $\delta > 0$, if $u \in \mathbb{R}$ and $|u - y| < \delta$ then $|u^n - y^n| < \epsilon$. Hint: use induction and consider the identity

$$u^n - y^n = (u - y)(u^{n-1} + u^{n-2}y + \cdots + y^{n-1}).$$

Solution. I managed to finish this problem without induction. Here is my claim:

$$\boxed{\text{Given } y \in \mathbb{R}, n \in \mathbb{N}, \text{ and } \epsilon > 0, \text{ if } \delta < \min\left(1, \frac{\epsilon}{n(|y|+1)^{n-1}}\right), \text{ then if } u \in \mathbb{R}, |u - y| < \delta \implies |u^n - y^n| < \epsilon.}$$

Proof. For convenience, we prefer to limit $\delta < 1$ using a min() function when necessary.[2]

If $|u - y| < 1$, then $|u| < |y| + 1$ and $|y| < |y| + 1$. It follows that

$$\begin{aligned} |u^n - y^n| &\leqslant |u - y| \left| \sum_{i=0}^{n-1} y^i u^{(n-1)-i} \right| && \text{(by the identity given in question)} \\ &\leqslant |u - y| \sum_{i=0}^{n-1} \left| y^i u^{(n-1)-i} \right| && \text{(iterations of triangle inequality)} \\ &< |u - y| \sum_{i=0}^{n-1} (|y| + 1)^{n-1} && \text{(assumption that } |y| < |y| + 1) \\ &< \delta(n)(|y| + 1)^{n-1}. \end{aligned}$$

Therefore if we set $\delta < \min\left(1, \dfrac{\epsilon}{n(|y|+1)^{n-1}}\right)$ then

$$|u^n - y^n| < \frac{\epsilon}{n(|y|+1)^{n-1}}(n)(|y| + 1)^{n-1} = \epsilon$$

from which we conclude that the claim holds.

[2]Idea came a casual chat with Prof. Andrew Manion during his office hours.

□

Alternate Solution 1. This time we will actually use (strong)[3] induction as suggested by the hint. Let $\varphi(n)$ be the statement

Given $y \in \mathbb{R}, n \in \mathbb{N}, \epsilon > 0$, there exists $\delta > 0$ such that if $u \in \mathbb{R}$ and $\lvert u-y\rvert < \delta$ then $\lvert u^k - y^k\rvert < \epsilon$ for all integers $k \in [0,n]$.

Furthermore, δ can be defined as

$$\delta < \begin{cases} \min\left(\dfrac{\epsilon}{2n|y|^{n-1}}, \dfrac{1}{2n|y|^{n-1}}\right), |y| \geqslant 1 \\ \\ \min\left(\dfrac{\epsilon}{2n|y|^{n-1}}, \dfrac{1}{2n|y|}\right), |y| < 1. \end{cases}$$

Clearly $\varphi(1)$ is trivial. To see $\varphi(2)$ is true, consider the following:

$$\begin{aligned}
|u^2 - y^2| &\leqslant |u-y||u+y| \\
&= |u-y|\left|\sum_{i=0}^{1} y^i u^{1-i}\right| && \text{(just written in another form)} \\
&\leqslant |u-y|\sum_{i=0}^{1} |y^i||u^{1-i}| \\
&\leqslant |u-y|\sum_{i=0}^{1}\left[|y^i|\left(|y^{1-i}| + |u^{1-i} - y^{1-i}|\right)\right] && \text{(triangle inequality)} \\
&< |u-y|\sum_{i=0}^{1}\left[|y^i|\left(|y^{(1-i)}| + \epsilon\right)\right] && \text{(by the assumption that } |u^k - y^k| < \epsilon \text{ for all } k \in [0,2]) \\
&\leqslant |u-y|\sum_{i=0}^{1}\left(|y| + \epsilon|y^i|\right) \\
&\leqslant |u-y|\left(2|y| + 2\epsilon|y|\right) < \delta\left(2|y| + 2\epsilon|y|\right).
\end{aligned}$$

Honestly, the case for $n = 2$ could have been a lot simpler. However I deliberately did so just to illustrate that the definition of δ above works. If $n = 2$ then

$$\delta < \min\left(\frac{\epsilon}{4|y|}, \frac{1}{4|y|}\right) \text{ regardless of the magnitude of } y.$$

Substituting these values into the equation we see

$$\text{If } \frac{\epsilon}{4|y|} < \frac{1}{4|y|}, \text{then } |u^2 - y^2| < \frac{\epsilon}{4|y|}(2|y|) + \frac{\epsilon}{4|y|}(2\epsilon|y|) < \frac{\epsilon}{4|y|}(2|y|) + \frac{1}{4|y|}(2\epsilon|y|) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ and}$$
$$\text{If } \frac{1}{4|y|} \leqslant \frac{\epsilon}{4|y|}, \text{then } |u^2 - y^2| < \frac{1}{4|y|}(2|y|) + \frac{1}{4|y|}(2\epsilon|y|) \leqslant \frac{1}{4|y|}(2|y|) + \frac{\epsilon}{4|y|}(2\epsilon|y|) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\varphi(2)$ is also true. Now assume $\varphi(k)$ is true, and we expand $|u^{k+1} - y^{k+1}|$ below to show $\varphi(k+1)$ also holds:

[3]Linfeng suggested that I try strong induction as opposed to weak induction, and he gave me hints on how to set up $\varphi(n)$.

$$\begin{aligned}
|u^{k+1} - y^{k+1}| &= \left|(u-y)\left(\sum_{i=0}^{k} y^i u^{k-i}\right)\right| && \text{(by the provided identity)}\\
&\leqslant |u-y| \sum_{i=0}^{k} \left[|y^i||u^{k-i}|\right]\\
&\leqslant |u-y| \sum_{i=0}^{k} \left[|y^i|\left(|y^{k-i}| + |u^{k-i} - y^{k-i}|\right)\right] && \text{(triangle inequality)}\\
&< |u-y| \sum_{i=0}^{k} \left[|y^k| + \epsilon|y^i|\right] && \text{(induction hypothesis, all } |u^i - y^i| < \epsilon)\\
&\leqslant |u-y| \left(\sum_{i=0}^{k} |y|^n + \sum_{i=0}^{k} \max(|y|, |y|^k)\cdot\epsilon\right) && \left(\text{maximizing } \sum_{i=0}^{k} \epsilon|y^i|\right)\\
&= \delta(k+1)|y|^k + \delta\epsilon(k+1)\max(|y|, |y|^k)
\end{aligned}$$

Note that since we don't know if $|y| \geqslant 1$, we don't know if $|y|$ or $|y|^k$ is the largest element in the sequence $|y|, |y|^2, \ldots, |y|^k$. Therefore we have to use a max() function to determine so. Since the formula for δ is highly similar in both cases, we will only verify the former here. Suppose $|y| \geqslant 1$, $n = k+1$, and we take

$$\delta < \min\left(\frac{\epsilon}{2(k+1)|y|^k}, \frac{1}{2(k+1)|y|^k}\right).$$

If $\dfrac{\epsilon}{2(k+1)|y|^k} < \dfrac{1}{2(k+1)|y|^k}$, then

$$|u^{k+1} - y^{k+1}| < \frac{\epsilon}{2(k+1)|y|^k}(k+1)|y|^k + \frac{\epsilon}{2(k+1)|y|^k}\epsilon(k+1)|y|^k < \frac{\epsilon}{2} + \frac{1}{2(k+1)|y|^k}\epsilon(k+1)|y|^k = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and if $\dfrac{1}{2(k+1)|y|^k} \leqslant \dfrac{\epsilon}{2(k+1)|y|^k}$, then

$$|u^{k+1} - y^{k+1}| < \frac{1}{2(k+1)|y|^k}(k+1)|y|^k + \frac{1}{2(k+1)|y|^k}\epsilon(k+1)|y|^k \leqslant \frac{\epsilon}{2(k+1)|y|^k}(k+1)|y|^k + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore we've shown $\varphi(k) \implies \varphi(k+1)$.

Having completed the base cases and the inductive step, we claim $\varphi(n)$ holds for all $n \in \mathbb{N}$. □

Alternate Solution 2. Another solution uses weak induction but doesn't use the identity provided. However it's much less complicated. Before coming up with this solution, I was wondering if it's possible to express the degree $n+1$ function $|u^{n+1} - y^{n+1}|$ by only using degree 1 and degree n functions — specifically $|u^n - y^n|$. The answer turns out to be very elegant:

$$\begin{aligned}
|u^{n+1} - y^{n+1}| &\leqslant |u^{n+1} - uy^n| + |uy^n - y^{n+1}|\\
&\leqslant |u||u^n - y^n| + |y^n||u-y|
\end{aligned} \tag{1}$$

Similar to what we've done in the first proof, we may well keep $|u - y| < 1$ using a min() function so $|u| < |y| + 1$. Let $\varphi(n)$ be the statement that

$$\boxed{\text{Given } y \in \mathbb{R}, n \in \mathbb{N}, \text{ and } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } u \in \mathbb{R}, |u-y| < \delta \implies |u^n - y^n| < \epsilon.}$$

Again $\varphi(1)$ is trivial. To show $\varphi(2)$ holds:

$$\begin{aligned} |u^2 - y^2| &\leqslant |u^2 - uy| + |uy - y^2| \\ &\leqslant |u||u-y| + |y||u-y| \\ &\leqslant (|y|+1)|u-y| + |y||u-y| \\ &< \delta(2|y|+1) \end{aligned}$$

Therefore if we set $\delta < \min\left(1, \frac{\epsilon}{2|y|+1}\right)$ then $|u^2 - y^2| < \frac{\epsilon}{2|y|+1}(2|y|+1) = \epsilon$. Now, for the inductive step, if we assume $\varphi(k)$ to be true, then there exists δ_k such that if $|u-y| < \delta_k$ then $|u^k - y^k| < \frac{\epsilon}{2(|y|+1)}$. (This fraction is particularly useful because we've set $|u| \leqslant |y|+1$ and the $(|y|+1)$ would cancel each other out during multiplication, leaving us with a neat $\epsilon/2$.) If we let

$$\delta_{k+1} < \min\left(1, \delta_k, \frac{\epsilon}{2(|y|^k+1)}^{\dagger}\right)$$

†2 to create $\epsilon/2$, $|y|^k$ to cancel out $|y^k|$ in inequality (1), +1 to avoid zero denominator.

then

$$\begin{aligned} |u^{k+1} - y^{k+1}| &\leqslant |u||u^k - y^k| + |y^k||u-y| && \text{(by the inequality (1) above)} \\ &< (|y|+1) \underbrace{\frac{\epsilon}{2(|y|+1)}}_{(<|u^k-y^k| \text{ by hypothesis})} + (|y^k|+1) \underbrace{\frac{\epsilon}{2(|y|^k+1)}}_{\text{(by the construction of } \delta_{k+1})} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore $\varphi(k) \implies \varphi(k+1)$ and the induction is done. Hence question proven. □

Problem 6 (1.16)**.** Given $x > 0$ and $n \in \mathbb{N}$, prove that there is a unique $y > 0$ such that $y^n = x$. That is, the n^{th} root of x exists and is unique. Hint: consider $y = \sup(\mathcal{S})$ where $\mathcal{S} := \{s \in \mathbb{R} \mid s^n \leqslant x\}$ and then show that y^n can neither $< x$ or $> x$.

Solution. (I'm not sure if it's necessary to show the existence of $\sup(\mathcal{S})$ for this question, since the hint already takes it as granted. Anyway, it's clear that $0^n = 0 < x$ and $(x+1)^n \geqslant x+1 > x$. Therefore $\mathcal{S}$ is nonempty and bounded from above, and we may safely proceed to assume $y = \sup(\mathcal{S})$.)[4]

As suggested by the hint, the proof of existence divides into two parts. We first suppose, by contradiction, that $y^n < x$, and we pick $\epsilon < x - y^n$ (so $y^n + \epsilon < x$). Now, by the conclusion of the previous problem, we know that

$$\delta < \frac{\epsilon}{n(y+\delta)^{n-1}} \implies (y+\delta)^n - y^n < \epsilon.$$

[4]Idea comes from a casual chat with Jiayue who believes that it's necessary to show the existence of $\sup(\mathcal{S})$.

(Note that we no longer need to use $(|y|+1)^{n-1}$ because we know that 1) both y and $y+\delta$ are positive and therefore their absolute values are the same as themselves and 2) we know $\max(y, y+\delta) = y+\delta$.) Therefore

$$(y+\delta)^n < y^n + \epsilon < x$$

from which we can immediately tell that $y+\delta \in \mathcal{S}$. Therefore y cannot be $\sup(\mathcal{S})$ because it's not even an upper bound of $\mathcal{S}$.

Now for the other part, suppose $y^n > x$ and we pick $\epsilon < y^n - x$ (so that $y^n - \epsilon > x$). Similarly, by the conclusion of the previous problem, we have

$$\delta < \frac{\epsilon}{ny^{n-1}} \implies y^n - (y-\delta)^n < \epsilon.$$

Therefore

$$(y-\delta)^n > y^n - \epsilon > x$$

from which the existence of $y-\delta$ shows y is not the L.U.B. of $\mathcal{S}$. Hence it's impossible that $y^n > x$.

By trichotomy, $y^n \nless x$ and $y^n \ngtr x$ implies $y^n = x$.

Now we try to prove the uniqueness of such y. Two proofs below, both supposes that $y' \in \mathbb{R}$ and $(y')^n = x$.

First proof. Clearly y' is also an upper bound of $\mathcal{S}$. It's also clear that y' is the L.U.B. because for any $z \in \mathbb{R}$, $z < y' \implies z^n < x$ and thus z cannot be an upper bound of $\mathcal{S}$ as shown above. By trichotomy exactly one among $y < y', y = y', y > y'$ is true. If $y < y'$ then y' isn't the *least* upper bound, whereas if $y > y'$ then y isn't the *least* upper bound. Therefore the only possibility is if $y = y'$. Hence proven. □

Second proof. We will first need a lemma.

Lemma. If $0 < a < b$, then $a^n < b^n$ for all $n \in \mathbb{Z}^+$.

Proof of lemma. Suppose $0 < a < b$. Let $\varphi(n)$ be the statement that $a^n < b^n$. Clearly $\varphi(1)$ is true. Now suppose $\varphi(k)$ holds, i.e., $a^k < b^k$. Then since $\mathbb{R}$ is a well-ordered field, we have $a^{k+1} < b \cdot a^{k+1} < b^{k+1}$. □

Back to the question. By trichotomy, if $y' \neq y$, then either $y < y'$ or $y' < y$, and either $x = y^n < (y')^n$ or $(y')^n < y^n = x$. Therefore if $(y')^n = x$ then $y' = y$, i.e., y is unique.

□