MATH 425a Problem Set 3

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Problem 1 (1.18)**.** Prove that real numbers correspond bijectively to decimal expansions not terminating in an infinite strings of nines, as follows. The decimal expansion of $x \in \mathbb{R}$ is $N.x_1x_2\ldots$, where N is the largest integer $\leqslant x$, x_1 is the largest integer $\leqslant 10(x-N)$, x_2 is the largest integer $\leqslant 100(x-(N+x_1/10))$, and so on.

(1) Show that each x_k is a digit between 0 and 9.

(2) Show that for each k there is an $\ell \geqslant k$ such that $x_\ell \neq 9$.

(3) Conversely, show that for each such expansion $N.x_1x_2\ldots$ not terminating in an infinite string of nines, the set

$$\{N, N+\frac{x_1}{10}, N+\frac{x_1}{10}+\frac{x_2}{100}, \ldots\}$$

is bounded and its least upper bound is a real number x with decimal expansion $N.x_1x_2\ldots$.

(4) Repeat the exercise with a general base in place of 10.

Solution.

(1) We will prove $0 \leqslant x_k \leqslant 9$ for all k by induction. Let $\varphi(n)$ be the statement that $0 \leqslant x_n \leqslant 9$ (alternatively, since all digits are integers, $0 \leqslant x_n < 10$) and that

$$0 \leqslant 10^n\left(x-\left(N+\frac{x_1}{10}+\cdots+\frac{x_n}{10^n}\right)\right) < 1.$$

First let's check $\varphi(1)$. Since N is the greatest integer not exceeding x, we have $0 \leqslant x-N < 1$. Therefore $0 \leqslant 10(x-N) < 10$. It follows that $0 \leqslant x_1 \leqslant 9$ since x_1 is the largest integer not exceeding $10(x-N)$ by definition. From this definition of x_1 we also have

$$0 \leqslant 10(x-N)-x_1 = 10\left(x-\left(N+\frac{x_1}{10}\right)\right) < 1.$$

Both conditions are met, and $\varphi(1)$ holds.

Now we assume $\varphi(m)$ is true. By definition of x_{m+1} and the induction hypothesis, we have

$$\begin{cases} (1)\ x_{m+1} \text{ is the largest integer } \leqslant 10^{m+1}\left(x-\left(N+\frac{x_1}{10}+\cdots+\frac{x_m}{10^m}\right)\right) \\ (2)\ 0 \leqslant 10^m\left(x-\left(N+\frac{x_1}{10}+\cdots+\frac{x_m}{10^m}\right)\right) < 1 \implies 0 \leqslant 10^{m+1}\left(x-\left(N+\frac{x_1}{10}+\cdots+\frac{x_m}{10^m}\right)\right) < 10 \end{cases}$$

from which it becomes clear that $0 \leqslant x_{m+1} \leqslant 9$ and

$$0 \leqslant 10^{m+1}\left(x - \left(N + \frac{x_1}{10} + \cdots + \frac{x_m}{10^m}\right)\right) - x_{m+1} = 10^{m+1}\left(x - \left(N + \frac{x_1}{10} + \cdots + \frac{x_m}{10^m} + \frac{x_{m+1}}{10^{m+1}}\right)\right) < 1.$$

Therefore $\varphi(m) \implies \varphi(m+1)$, and we conclude that each x_k is a digit between 0 and 9.

(2) Before proving this part, we need to introduce a lemma.

Lemma. The infinite sum $\sum_{i=1}^{\infty} \frac{9}{10^i} = \frac{9}{10} + \frac{9}{10^2} + \dots$ is equal to 1.

Proof of lemma. Suppose $\sum_{i=1}^{\infty} \frac{9}{10^i} = S$. If we multiply each term by 10, we get a new geometric series with infinite sum $10S$. Subtracting these two gives

$$\begin{aligned} 10S - S = 9S &= \sum_{i=1}^{\infty} \frac{9}{10^{i-1}} - \sum_{i=1}^{\infty} \frac{9}{10^i} \\ &= \left(9 + \frac{9}{10} + \frac{9}{10^2} + \dots\right) - \left(\frac{9}{10} + \frac{9}{10^2} + \dots\right) \\ &= 9 \end{aligned}$$

Hence $9S = 9, S = 1$, which finishes the proof. □

Back to the question. Suppose, for contradiction, that there exists a k such that $x_\ell = 9$ for all $\ell \geqslant k$. Recall that x_{k-1} is the largest integer not exceeding $10^{k-1}\left(x - \left(N + \frac{x_1}{10} + \cdots + \frac{x_{k-2}}{10^{k-2}}\right)\right)$. Observe that, based on the definition of x_ℓ, it's always true that $x - \left(N + x_1/10 + \cdots + x_\ell/10^\ell\right)$ is nonnegative. Therefore,

$$x - \left(N + \frac{x_1}{10} + \cdots + \frac{x_{k-1}}{10^{k-1}} + \frac{9}{10^k} + \frac{9}{10^{k+1}} + \dots\right) \geqslant 0$$

By the lemma, the inequality above can be re-written as

$$x - \left(N + \frac{x_1}{10} + \cdots + \frac{x_{k-1}}{10^{k-1}} + \frac{1}{10^{k-1}}\right) = x - \left(N + \frac{x_1}{10} + \cdots + \frac{x_{k-2}}{10^{k-2}} + \frac{x_{k-1}+1}{10^{k-1}}\right) \geqslant 0$$

which tells us that $x_{k-1} + 1$ is also an integer not exceeding $10^{k-1}\left(x - \left(N + \frac{x_1}{10} + \cdots + \frac{x_{k-2}}{10^{k-2}}\right)\right)$. Contradiction. Hence there cannot exist a non-terminating string of 9 at the end of a decimal expansion.

(3) For convenience let us denote this set as $\mathcal{S}$. Clearly $\mathcal{S}$ is nonempty, and since $N.x_1x_2\dots$ is not terminating in an infinite string of 9's, $\mathcal{S}$ is bounded above by

$$N + \frac{9}{10} + \frac{9}{10^2} + \cdots = N + \sum_{i=1}^{\infty} \frac{9}{10^i} = N + 1.$$

It follows that the L.U.B. property applies, and we may denote $x = \sup(\mathcal{S})$. From above we see $x \leqslant N + 1$. Again, since the x_k's are not a string of non-terminating 9's, $x \neq N + 1$. Also note that each $x_k/10^k$ is nonnegative, so $\mathcal{S}$ is bounded below by N. Therefore the integer part of x must be N.

Now we start working on x_1. Since

$$x = \sup\left\{N, N + \frac{x_1}{10}, N + \frac{x_1}{10} + \frac{x_2}{100}, \dots\right\},$$

we also have

$$10(x - N) = \sup\left\{x_1, x_1 + \frac{x_2}{10}, x_1 + \frac{x_2}{10} + \frac{x_3}{100}, \dots\right\}.$$

Just like $\mathcal{S}$, this new set is nonempty and bounded below by x_1 and above by $x_1 + 1$. Furthermore, just like $x < N + 1$, $10(x - N) < x_1 + 1$ because $x_2, x_3, \dots$ is not a string of non-terminating 9's. Therefore the largest integer not exceeding $10(x - N)$ is indeed x_1, and we've shown that it is also the first digit of x after the decimal point.

We could have set up an induction to show that the k^{th} digit of x after the decimal point is indeed x_k, but illustrating by example is easier to follow. The L.U.B. of $\left\{x_k, x_k + \frac{x_{k+1}}{10}, x_k + \frac{x_{k+1}}{10} + \frac{x_{k+2}}{100} \dots\right\}$ is greater than x_k and strictly less than $x_k + 1$ as $N.x_1x_2\dots$ does not contain a non-terminating string of 9's.

Hence $x = N.x_1x_2\cdots = \sup(\mathcal{S})$.

(4) The steps are almost identical and, for base n, we simply need to change the lemma to

$$\sum_{i=1}^{\infty} \frac{n-1}{n^i} = \frac{n-1}{n} + \frac{n-1}{n^2} + \cdots = 1.$$

Then, the base n decimal expansion of $x \in \mathbb{R}$ that does not have a never-ending string of $(n-1)$ is $N.n_1n_2\dots$ where N is the largest integer not exceeding x, n_1 the largest integer not exceeding $n(x - N)$, and n_k the integer not exceeding $n^k\left(x - \left(N + \sum_{i=1}^{k-1} \frac{n_i}{n^i}\right)\right)$. The corresponding three parts become

(1) each n_k is a digit between 0 and $n - 1$.

(2) for each k there exists an $\ell \geqslant k$ such that $n_\ell \neq n - 1$.

(3) for each expansion $N.n_1n_2\dots$ not terminating in an infinite string of $(n-1)$'s, the set

$$\left\{N, N + \frac{n_1}{n}, N + \frac{n_1}{n} + \frac{n_2}{n^2}, \dots\right\}$$

is bounded and its supremum is precisely the real number x with decimal expansion $N.n_1n_2\dots$ (in base n).

Problem 2 (1.19)**.** Formulate the definition of the **greatest lower bound** of a set of real numbers. State a G.L.B. property of $\mathbb{R}$ and show it is equivalent to the L.U.B. property of $\mathbb{R}$.

Solution. Greatest Lower Bound Property:

If $\mathcal{S}$ is a nonempty subset of $\mathbb{R}$ and is bounded below in $\mathbb{R}$ then $\exists$ a G.L.B. for $\mathcal{S}$

where a G.L.B. is an element $x \in \mathbb{R}$ such that

(1) $x < s$ for all $s \in \mathcal{S}$ and

(2) for all $y \in \mathbb{R}$ satisfying $y < s$ for all $s \in S$, $x \geqslant y$.

I am not sure what the question means by asking me to "show [the G.L.B.] property is equivalent to the L.U.B. property". Clearly to show that two propositions are equivalent we need to to show L.U.B. $\implies$ G.L.B. and vice versa. For the forward direction, one interpretation is to show that if $\mathcal{S}$ is nonempty and bounded above then $\sup(\mathcal{S}) = -\inf(-\mathcal{S})$. Another way is to show that if $x = \sup(\mathcal{S})$ then $x = \inf(\mathcal{T})$ for a set $\mathcal{T}$ that is nonempty and bounded below. I will do both here, but please also read the paragraph labeled † below.

(1) For the first interpretation, we have shown the forward direction in class. The box below is a screenshot:

> Since $\mathcal{S}$ is nonempty and bounded from below, we know $(-\mathcal{S})$ is nonempty and bounded from above. Therefore $(-\mathcal{S})$ has a L.U.B. Suppose $\sup(-\mathcal{S}) = b$. Claim: $-b = \inf(\mathcal{S})$.
>
> First show that $-b$ is a lower bound. Since $b \geqslant -s, \forall s \in \mathcal{S}$, we know $-b \leqslant s, \forall s \in \mathcal{S}$. Therefore $-b$ is a lower bound for $\mathcal{S}$.
>
> Now we show that it's the greatest among all lower bounds. Let $-b'$ be another lower bound. By the same argument b' is also an upper bound for $(-\mathcal{S})$. Since $-b = \sup(\mathcal{S})$, it follows that $b \leqslant b'$, and $-b \geqslant -b'$. Therefore $-b$ is indeed $\inf(\mathcal{S})$.

From discussion on Tue, 9/1

For G.L.B. $\implies$ L.U.B., suppose $x = \inf(\mathcal{S})$ and define $(-\mathcal{S}) = \{-s \mid s \in \mathcal{S}\}$. By definition, $x \leqslant s$ for all $s \in \mathcal{S}$. Therefore $-x \geqslant -s$ for all $-s \in (-\mathcal{S})$. Therefore $-x$ is an upper bound for $(-\mathcal{S})$. Now let $-y$ also be an upper bound for $(-\mathcal{S})$, and we have $-y \geqslant -x$ for all $-x \in (-\mathcal{S})$. Negating both sides we have $y \leqslant x$ for all $x \in \mathcal{S}$. Therefore y is a lower bound for $\mathcal{S}$. Since $x = \inf(\mathcal{S})$ we know $y \leqslant x$. Therefore $-y \geqslant -x$ and $-x$ is indeed $\sup(-\mathcal{S})$.

(2) For the second interpretation, first assume $\mathcal{S}$ is a nonempty set bounded below and assume the existence of the L.U.B. property. Now consider the set $\mathcal{T} = \{t \in \mathbb{R} \mid t \leqslant s \ \forall s \in \mathcal{S}\}$. Since $\mathcal{S}$ is bounded below, $\mathcal{T}$ is nonempty. Clearly $\mathcal{T}$ is also bounded above by any $s \in \mathcal{S}$. Therefore the L.U.B. property applies, and there exists $t^* = \sup(\mathcal{T})$. It follows that, in addition to each $s \in \mathcal{S}$ being an upper bound for $\mathcal{T}$, each $t \in \mathcal{T}$ is also a lower bound for $\mathcal{S}$. Therefore t^* is not only the L.U.B. of $\mathcal{T}$ but also the greatest among all lower bounds for $\mathcal{S}$, i.e., $t^* = \sup(\mathcal{T}) = \inf(\mathcal{S})$.

Similarly, we may assume $\mathcal{T}$ is a nonempty set and assume that the G.L.B. property exists. Then if we consider the set $\mathcal{S} = \{s \in \mathbb{R} : s \geqslant t \ \forall t \in \mathcal{T}\}$ and apply the G.L.B. property to $\mathcal{S}$, we will reach the similar conclusion that $\sup(\mathcal{T}) = \inf(\mathcal{S})$.

† *I don't see much difference between the two approaches. Both start by assuming the existence of L.U.B. property (or G.L.B.) of $\mathbb{R}$ and show that the G.L.B. property (or L.U.B.) applies to some subset of $\mathbb{R}$. If you believe the second one doesn't make much sense, please just ignore it.*

Problem 3 (1.20)**.** Prove that limits are unique, i.e., if (a_n) is a sequence of real numbers that converges to a real number b and also converges to a real number b', then $b = b'$.

Solution. Suppose, by contradiction, that (a_n) converges to both b and b' with $b \neq b'$. Then we may set $\epsilon = |b - b'|$. Since the sequence converges to b, there exists $N \in \mathbb{Z}^+$ such that $k \geqslant N \implies |a_k - b| < \epsilon/2$. Likewise, since the sequence also converges to b', there exists another $N' \in \mathbb{Z}^+$ such that $k \geqslant N' \implies |a_k - b'| < \epsilon/2$. Now if we set $N^* = \max(N, N')$, then if $k \geqslant N^*$ we have

$$|b - b'| \leqslant |b - a_k| + |a_k - b'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

contradicting the assumption $|b - b'| = \epsilon$. Therefore (a_n) can't converge to two distinct limits, i.e., limits are unique.

Problem 4 (1.27)**.** Prove that the interval $[a, b] \in \mathbb{R}$ is the same as the segment $[a, b] \in \mathbb{R}^1$. That is,

$$\begin{aligned} &\{x \in \mathbb{R} : a \leqslant x \leqslant b\} \\ =&\{y \in \mathbb{R} : \exists s, t \in [0, 1] \text{ with } s + t = 1 \text{ and } y = sa + tb\}. \end{aligned}$$

Solution. For convenience, let us denote the first set (interval) by A and the second (segment) by B. To show $A = B$ we need to show that A and B are mutually inclusive. Showing $B \subset A$ is relatively easier: for all s, t such that $s + t = 1$, we have

$$a = sa + ta < sa + tb < sb + tb = b \implies sa + tb \in [a, b].$$

For the converse, the metacognition here is that we want to create a *linear function* (we haven't defined it, but you know what I mean) that satisfies $f(a) = 0$, $f(b) = 1$, and $(f(x) - f(y))/(x - y)$ remains as a constant. Therefore we can consider the following:

$$\begin{cases} s = \dfrac{b - x}{b - a} \\ t = \dfrac{x - a}{b - a} \end{cases} \implies s + t = 1 \text{ and } f(x) = sa + tb = \frac{b - x}{b - a} a + \frac{x - a}{b - a} b = x.$$

This shows that for any $x \in [a, b]$, we are able to come up with s, t such that $s + t = 1$ and $x = sa + tb$. Hence $A \subset B$ and, together with $B \subset A$, we conclude that $A = B$.

Problem 5 (1.28)**.** A **convex combination** of $w_1, \ldots, w_k \in \mathbb{R}^m$ is a vector sum

$$w = s_1 w_1 + \cdots + s_k w_k$$

such that $s_1 + \cdots + s_k = 1$ and $0 \leqslant s_1, \ldots, s_k \leqslant 1$.

(1) Prove that if a set E is convex the E contains the convex combination of any finite number of points in E.

(2) Why is the converse obvious?

Solution.

(1) We will prove by induction. Let E be a convex set, and let $\varphi(n)$ be the statement that

$$\boxed{E \text{ contains all convex combinations of } n \text{ points in } E.}$$

Clearly $\varphi(1)$ is trivial, and $\varphi(2)$ is also trivial since E is convex. We may proceed to the inductive step now.

Now we assume $\varphi(k)$ is true, and we try to show $\varphi(k) \implies \varphi(k+1)$. Suppose we have $w_1, \dots, w_k, w_{k+1} \in \mathbb{R}^m$, and we want to show that $\lambda_1 w_1 + \cdots + \lambda_{k+1} w_{k+1} \in E$ as long as $\sum \lambda = 1$. Note that currently the sum of all coefficients excluding λ_{k+1} is $1 - \lambda_{k+1}$. If we focus on the first k terms and let $s_i = \lambda_i/(1-\lambda_{k+1})$, then $\sum_{i=1}^k s_i = (1-\lambda_{k+1})/(1-\lambda_{k+1}) = 1$. Therefore, by the induction hypothesis, the convex combinations of the first k vectors is also in E, i.e.,

$$\mathbf{w} = \frac{\lambda_1}{1-\lambda_{k+1}} w_1 + \frac{\lambda_2}{1-\lambda_{k+1}} w_2 + \cdots + \frac{\lambda_k}{1-\lambda_{k+1}} w_k = \sum_{i=1}^{k} \frac{\lambda_i w_i}{1-\lambda_{k+1}} \in E.$$

Then, our arbitrary convex combination $\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_{k+1} w_{k+1}$ becomes

$$(\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_k w_k) + \lambda_{k+1} w_{k+1} = (1-\lambda_{k+1})\mathbf{w} + \lambda_{k+1} w_{k+1},$$

a convex combination of merely two vectors in E. Since E is convex, this combination is also in E. Hence $\varphi(k+1)$ holds, and we are done with the proof. □

(2) Because the converse doesn't require $\varphi(n)$ to be true for all $n \in \mathbb{N}$: $\varphi(2)$ alone is already sufficient to show that E is convex.

Problem 6 (1.29 (a))**.** Prove that the ellipsoid

$$E = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1 \right\}$$

is convex. [Hint: E is the unit ball for a different dot product.]

Solution. First I will provide two ways to define an inner product. (Why not when we can have a bit of fun?)

(1) We can rewrite the equation of an ellipsoid in matrix form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1/a^2 & 0 & 0 \\ 0 & 1/b^2 & 0 \\ 0 & 0 & 1/c^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{x}^T Q \mathbf{x} \leqslant 1.$$

To see $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T Q \mathbf{w}$ is an inner product, we verify its symmetry, linearity, and positive definiteness:

(I) Symmetry: $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T Q \mathbf{w} = (\mathbf{v}^T Q \mathbf{w})^T = \mathbf{w}^T Q^T (\mathbf{v}^T)^T = \mathbf{w}^T Q \mathbf{v} = \langle \mathbf{w}, \mathbf{v} \rangle$.

(II) Linearity: $\langle c\mathbf{v}, \mathbf{w} \rangle = (c\mathbf{v})^T Q \mathbf{w} = c(\mathbf{v}^T Q \mathbf{w}) = c\langle \mathbf{v}, \mathbf{w} \rangle$ and
$\langle \mathbf{v} + \mathbf{v}', \mathbf{w} \rangle = (\mathbf{v} + \mathbf{v}')^T Q \mathbf{w} = \mathbf{v}^T Q \mathbf{w} + (\mathbf{v}')^T Q \mathbf{w} = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{v}', \mathbf{w} \rangle$.

(III) Positive definiteness: $\langle \mathbf{v}, \mathbf{v} \rangle = \mathbf{v}^T Q \mathbf{v}$ is positive definite because the eigenvalues of Q are $1/a^2, 1/b^2$, and $1/c^2$, all of which are positive.

(2) Alternatively, given $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$,we can consider the map $\langle \mathbf{x}, \mathbf{y} \rangle \to \mathbb{R}$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{x_1 y_1}{a^2} + \frac{x_2 y_2}{b^2} + \frac{x_3 y_3}{c^2}.$$

Again, to see whether this is an inner product, we must check its symmetry, linearity, and positive definiteness:

(I) Symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \frac{x_1 y_1}{a^2} + \frac{x_2 y_2}{b^2} + \frac{x_3 y_3}{c^2} = \frac{y_1 x_1}{a^2} + \frac{y_2 x_2}{b^2} + \frac{y_3 x_3}{c^2} = \langle \mathbf{y}, \mathbf{x} \rangle$.

(II) Linearity: $\langle c\mathbf{x}, \mathbf{y} \rangle = \frac{cx_1 y_1}{a^2} + \frac{cx_2 y_2}{b^2} + \frac{cx_3 y_3}{c^2} = c\left(\frac{x_1 y_1}{a^2} + \frac{x_2 y_2}{b^2} + \frac{x_3 y_3}{c^2}\right) = c \langle \mathbf{x}, \mathbf{y} \rangle$ and

$\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \frac{(x_1 + z_1) y_1}{a^2} + \frac{(x_2 + z_2) y_2}{b^2} + \frac{(x_3 + z_3) y_3}{c^2} = \frac{x_1 y_1 + z_1 y_1}{a^2} + \frac{x_2 y_2 + z_2 y_2}{b^2} + \frac{x_3 y_3 + z_3 y_3}{c^2} = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$.

(III) Positive definiteness: $\langle \mathbf{x}, \mathbf{x} \rangle = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \geqslant 0$ since the numerators are all nonnegative and the denominators are all positive.

In either case, the ellipsoid E is the unit ball with norm $\leqslant 1$. To show it's convex, suppose $\mathbf{v}, \mathbf{w} \in E$. It follows that $0 < \|\mathbf{v}\|, \|\mathbf{w}\| \leqslant 1$. We want to show that $\lambda \mathbf{v} + (1 - \lambda)\mathbf{w} \in E$ for all $\lambda \in [0, 1]$, i.e., its norm $\leqslant 1$. Since

$$\begin{aligned}
\langle \lambda \mathbf{v} + (1 - \lambda)\mathbf{w}, \lambda \mathbf{v} + (1 - \lambda)\mathbf{w} \rangle &= \lambda^2 \langle \mathbf{v}, \mathbf{v} \rangle + 2\lambda(1 - \lambda) \langle \mathbf{v}, \mathbf{w} \rangle + (1 - \lambda)^2 \langle \mathbf{w}, \mathbf{w} \rangle && \text{(applying linearity)} \\
&= \lambda^2 \|\mathbf{v}\| + (1 - \lambda)^2 \|\mathbf{w}\| + 2\lambda(1 - \lambda) \langle \mathbf{v}, \mathbf{w} \rangle && \text{(definition of norm)} \\
&\leqslant \lambda^2 \|\mathbf{v}\| + (1 - \lambda)^2 \|\mathbf{w}\| + 2\lambda(1 - \lambda) \|\mathbf{v}\| \|\mathbf{w}\| && \text{(Cauchy-Schwarz inequality)} \\
&\leqslant \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda) && (\|\mathbf{v}\|, \|\mathbf{w}\| \leqslant 1\ ) \\
&= 1
\end{aligned}$$

we conclude that the ellipsoid is indeed convex. □

Problem 7 (1.29 (b))**.** Prove that all boxes in $\mathbb{R}^m$ are convex.

Solution. All boxes in $\mathbb{R}^m$ have the form

$$[a_1, b_1] \times \cdots \times [a_m, b_m]$$

Suppose $\mathbf{x}, \mathbf{y} \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, then $a_i \leqslant x_i, y_i \leqslant b_i$. It follows that if $0 \leqslant \lambda \leqslant 1$, then

$$a_i \leqslant \min(x_i, y_i) \leqslant \lambda \min(x_i, y_i) + (1 - \lambda) \max(x_i, y_i) \leqslant \max(x_i, y_i) \leqslant b_i,$$

from which we see that any convex combinations of two points in the box produce another point in the box. Hence all boxes in $\mathbb{R}^m$ are convex.